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The decomposition of global conformal invariants IV: A proposition on local Riemannian invariants ☆,☆☆

Spyros Alexakis

University of Toronto, Toronto, Canada

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Abstract

This is the fourth in a series of papers where we prove a conjecture of Deser and Schwimmer regarding the algebraic structure of "global conformal invariants"; these are defined to be conformally invariant integrals of geometric scalars. The conjecture asserts that the integrand of any such integral can be expressed as a linear combination of a local conformal invariant, a divergence and of the Chern–Gauss–Bonnet integrand.

The present paper lays out the second half of this entire work: The second half proves certain purely algebraic statements regarding local Riemannian invariants; these were used extensively in the first two papers in this series, see Alexakis (2007, 2009) [2,3]. These results may be of independent interest, applicable to related problems.

Keywords: Conformal geometry; Local Riemannian invariants; Differential invariants

Contents

☆ This work has absorbed the best part of the author's energy over many years. This research was partially conducted during the period the author served as a Clay Research Fellow, an MSRI postdoctoral fellow, a Clay Liftoff fellow and a Procter Fellow.

☆☆ The author is immensely indebted to Charles Fefferman for devoting twelve long months to the meticulous proofreading of the present paper. He also wishes to express his gratitude to the Mathematics Department of Princeton University for its support during his work on this project.

E-mail address: alexakis@math.toronto.edu.

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1. Introduction

This is the fourth in a series of papers [2,4–7] where we prove a conjecture of Deser and Schwimmer [13] regarding the algebraic structure of global conformal invariants. We recall that a global conformal invariant is an integral of a natural scalar-valued function of Riemannian metrics, $\int_{M^n} P(g)\,dV_g$, with the property that this integral remains invariant under conformal re-scalings of the underlying metric.[1] More precisely, $P(g)$ is assumed to be a linear combination, $P(g)=\sum_{l\in L} a_l C^l(g)$, where each $C^l(g)$ is a complete contraction in the form:

$$contr^l\big(\nabla^{(m_1)}R\otimes\cdots\otimes\nabla^{(m_s)}R\big); \tag{1.1}$$

here each factor $\nabla^{(m)}R$ stands for the mth iterated covariant derivative of the curvature tensor R. ∇ is the Levi-Civita connection of the metric g and R is the curvature associated to this connection. The contractions are taken with respect to the quadratic form g^{ij}. In this series of papers we prove:

Theorem 1.1. *Assume that $P(g)=\sum_{l\in L} a_l C^l(g)$, where each $C^l(g)$ is a complete contraction in the form* (1.1)*, with weight $-n$. Assume that for every closed Riemannian manifold (M^n,g) and every $\phi\in C^\infty(M^n)$:*

$$\int_{M^n} P\big(e^{2\phi}g\big)\,dV_{e^{2\phi}g}=\int_{M^n} P(g)\,dV_g.$$

[1] See the introduction of [2] for a detailed discussion of the Deser–Schwimmer conjecture, and for background on scalar Riemannian invariants.

We claim that $P(g)$ can then be expressed in the form:

$$P(g) = W(g) + div_i T^i(g) + \mathrm{Pfaff}(R_{ijkl}).$$

Here $W(g)$ stands for a local conformal invariant of weight $-n$ (meaning that $W(e^{2\phi}g) = e^{-n\phi}W(g)$ for every $\phi \in C^\infty(M^n)$), $div_i T^i(g)$ is the divergence of a Riemannian vector field of weight $-n+1$, and $\mathrm{Pfaff}(R_{ijkl})$ *is the Pfaffian of the curvature tensor.*

We now discuss the position of the present paper in this series.

We recall from the introduction of [2] that this series of papers can be naturally subdivided into two parts: Part I (consisting of [2–4]) proves the Deser–Schwimmer conjecture, subject to establishing certain "main algebraic propositions", namely Proposition 5.2 in [2] and Propositions 3.1, 3.2 in [3]. Part II, consisting of the present paper and [6,7] prove these main algebraic propositions.

The first task that we undertake in the present paper is to reduce the "main algebraic propositions" in [2,3] to a single Proposition 2.1 below, which we call the "fundamental Proposition 2.1" and which will be proven by an elaborate induction on four parameters. This fundamental proposition is actually a *generalization* of the "main algebraic propositions" in [2,3], in the sense that the "main algebraic propositions" are special cases of Proposition 2.1; in fact they are the ultimate or penultimate steps of the aforementioned induction with respect to certain of the parameters.

An outline of the goals of the papers [5–7]: The main goal in the present paper is to introduce Proposition 2.1 below, which will imply the "main algebraic propositions" in [2,3], and then to reduce *the inductive step* in the proof of Proposition 2.1 to three Lemmas 3.1, 3.2, 3.5 below (along with two preparatory claims needed for Lemma 3.5, namely Lemmas 3.3, 3.4): We prove in the present paper that the three Lemmas 3.1, 3.2, 3.5 imply the inductive step of Proposition 2.1, apart from certain special cases. In this derivation we employ certain technical lemmas.[2] In the next paper in the series, [6] we derive the inductive step of Proposition 2.1 in the special cases, and we also provide a proof of the aforementioned technical lemmas. Thus, the present paper and [6] reduce the task of proving the Deser–Schwimmer conjecture to proving the Lemmas 3.1–3.5 below.

Then, Lemmas 3.1–3.5 are proven in the final paper in this series, [7].

Outline of the "fundamental Proposition 2.1": In Section 2 we set up the considerable notational and language conventions needed to state our fundamental Proposition 2.1; we then state the fundamental proposition and explain how the "main algebraic propositions" 5.2 and 3.1, 3.2 in [2,3] are special cases of it. We also explain how the fundamental proposition will be proven via an induction on four parameters. In Section 3 we distinguish three cases I, II, III on the hypothesis of Proposition 2.1 and claim three Lemmas 3.1, 3.2, 3.5 which correspond to these three cases. Finally in Section 4 we prove that these three lemmas *imply* Proposition 2.1.[3] In Section 4 we also assert certain important technical lemmas which will also be used in the subsequent papers in this series; some of these technical papers are proven in the present paper and some in [6].

Now, since the fundamental proposition is very complicated to even write out, we reproduce here the claim of the first "main algebraic proposition" from [2], for the reader's convenience.

[2] More on this in the outline of the present paper below.

[3] We make use of the inductive assumption of Proposition 2.1 in this derivation.

As explained above, this first "main algebraic proposition" is a special case of Proposition 2.1 below.

A simplified description of the main algebraic Proposition 5.2 in [2]: Given a Riemannian metric g over an n-dimensional manifold M^n and auxiliary C^∞ scalar-valued functions $\Omega_1, \ldots, \Omega_p$ defined over M^n, the objects of study are linear combinations of tensor fields $\sum_{l\in L} a_l C^{l,i_1\ldots i_\alpha}_g$, where each $C^{l,i_1\ldots i_\alpha}_g$ is a *partial contraction* with α free indices, in the form:

$$pcontr\big(\nabla^{(m)} R \otimes \cdots \otimes \nabla^{(m_s)} R \otimes \nabla^{(b_1)} \Omega_1 \otimes \cdots \otimes \nabla^{(b_m)} \Omega_p\big); \tag{1.2}$$

here $\nabla^{(m)} R$ stands for the mth covariant derivative of the curvature tensor R,[4] and $\nabla^{(b)} \Omega_h$ stands for the bth covariant derivative of the function Ω_h. A *partial contraction* means that we have list of pairs of indices $({}_a, {}_b), \ldots, ({}_c, {}_d)$ in (1.2), which are contracted against each other using the metric g^{ij}. The remaining indices (which are not contracted against another index in (1.2)) are the *free indices* ${}_{i_1}, \ldots, {}_{i_\alpha}$.

The "main algebraic proposition" of [2] (roughly) asserts the following: Let $\sum_{l\in L_\mu} a_l C^{l,i_1\ldots i_\mu}_g$ stand for a linear combination of partial contractions in the form (1.2), where each $C^{l,i_1\ldots i_\mu}_g$ has a given number σ_1 of curvature factors $\nabla^{(m)} R$ and a given number p of factor $\nabla^{(b)} \Omega_h$. Assume also that $\sigma_1 + p \geqslant 3$, each $b_i \geqslant 2$,[5] and that for each pair of contracting indices $({}_a, {}_b)$ in any given $C^{l,i_1\ldots i_\mu}_g$, the indices ${}_a, {}_b$ do not belong to the same factor. Assume also the rank $\mu > 0$ is fixed and each partial contraction $C^{l,i_1\ldots i_\mu}_g$, $l \in L_\mu$ has a given *weight* $-n+\mu$.[6] Let also $\sum_{l\in L_{>\mu}} a_l C^{l,i_1\ldots i_{y_l}}_g$ stand for a (formal) linear combination of partial contractions of weight $-n + y_l$, with all the properties of the terms indexed in L_μ, *except* that now all the partial contractions have a different rank y_l, and each $y_l > \mu$.

The assumption of the "main algebraic proposition" is a local equation:

$$\sum_{l\in L_\mu} a_l X\, div_{i_1} \ldots X\, div_{i_\mu} C^{l,i_1\ldots i_\mu}_g + \sum_{l\in L_{>\mu}} a_l X\, div_{i_1} \ldots X\, div_{i_{y_l}} C^{l,i_1\ldots i_{y_l}}_g = 0, \tag{1.3}$$

which is assumed to hold *modulo* complete contractions with $\sigma + 1$ factors. Here given a partial contraction $C^{l,i_1\ldots i_\alpha}_g$ in the form (1.2) $X\, div_{i_s}[C^{l,i_1\ldots i_\alpha}_g]$ stands for sum of $\sigma - 1$ terms in $div_{i_s}[C^{l,i_1\ldots i_\alpha}_g]$ where the derivative ∇^{i_s} is *not* allowed to hit the factor to which the free index ${}_{i_s}$ belongs.[7]

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[4] In other words it is an $(m+4)$-tensor; if we write out its free indices it would be in the form $\nabla^{(m)}_{r_1\ldots r_m} R_{ijkl}$.

[5] This means that each function Ω_h is differentiated at least twice.

[6] See [2] for a precise definition of weight.

[7] Recall that given a partial contraction $C^{l,i_1\ldots i_\alpha}_g$ in the form (1.2) with σ factors, $div_{i_s} C^{l,i_1\ldots i_\alpha}_g$ is a sum of σ partial contractions of rank $\alpha - 1$. The first summand arises by adding a derivative ∇^{i_s} onto the first factor T_1 and then contracting the upper index ^{i_s} against the free index ${}_{i_s}$; the second summand arises by adding a derivative ∇^{i_s} onto the second factor T_2 and then contracting the upper index ^{i_s} against the free index ${}_{i_s}$ etc.

The main algebraic proposition in [2] then claims that there will exist a linear combination of partial contractions in the form (1.2), $\sum_{h\in H} a_h C_g^{h,i_1\dots i_{\mu+1}}$ with all the properties of the terms indexed in $L_{>\mu}$, and all with rank $(\mu+1)$, so that:

$$\sum_{l\in L_\mu} a_l C_g^{l,(i_1\dots i_\mu)} + \sum_{h\in H} a_h X\, div_{i_{\mu+1}} C_g^{l,(i_1\dots i_\mu)i_{\mu+1}} = 0; \tag{1.4}$$

the above holds modulo terms of length $\sigma+1$. Also the symbol $(\dots)$ means that we are *symmetrizing* over the indices between parentheses.

A brief discussion of Proposition 2.1 and Lemmas 3.1, 3.2, 3.5: The fundamental Proposition 2.1 is a generalization of Proposition 5.2 from [2], in the sense that it deals with partial contractions in the form (1.2), which *in addition* contain factors $\nabla\phi_h$[8]; these are assumed to contract against the different factors $\nabla^{(m)}R$, $\nabla^{(p)}\Omega_x$ according to a given *pattern*.[9] Proposition 2.1 also groups up the different partial contractions indexed in L_μ according to the distribution of the free indices among its different factors $\nabla^{(m)}R$, $\nabla^{(p)}\Omega_x$.[10] The claim of Proposition 2.1 is then an adaptation of (1.4), restricted to a particular subset of the partial contractions indexed in L_μ. A discussion of how Proposition 2.1 is proven via an induction on four parameters, as well as how the inductive step is reduced to Lemmas 3.1, 3.2, 3.5 is provided in Subsections 3.1 and 3.2. The reader is also referred to those subsections for a more conceptual outline of the ideas in the present paper.

Before proceeding to give the strict formulation of the fundamental proposition, we digress to discuss the relationship of the whole work [2–7] and of the papers [5–7] in particular with the study of local scalar Riemannian and conformal invariants.

Broad discussion: The theory of *local* invariants of Riemannian structures (and indeed, of more general geometries, e.g. conformal, projective, or CR) has a long history. As discussed in [2], the original foundations of this field were laid in the work of Hermann Weyl and Élie Cartan, see [19,12]. The task of writing out local invariants of a given geometry is intimately connected with understanding polynomials in a space of tensors with given symmetries; these polynomials are required to remain invariant under the action of a Lie group on the components of the tensors. In particular, the problem of writing down all local Riemannian invariants reduces to understanding the invariants of the orthogonal group.

In more recent times, a major program was laid out by C. Fefferman in [14] aimed at finding all scalar local invariants in CR geometry. This was motivated by the problem of understanding the local invariants which appear in the asymptotic expansion of the Bergman and Szegö kernels of strictly pseudo-convex CR manifolds, in a similar way to which Riemannian invariants appear in the asymptotic expansion of the heat kernel; the study of the local invariants in the singularities of these kernels led to important breakthroughs in [9] and more recently by Hirachi in [17]. This program was later extended to conformal geometry in [15]. Both these geometries belong to a broader class of structures, the *parabolic geometries*; these admit a principal bundle with structure group a parabolic subgroup P of a semi-simple Lie group G, and a Cartan connection on that principle bundle (see the introduction in [10]). An important question in the study of these

[8] See the forms (2.1), (2.2) below.

[9] This encoding is described by the notions of *weak* and *simple* character—see the informal discussion after Definition 2.3.

[10] This encoding is described by the notions of *double* and *refined double* character—see the informal discussion after Definition 2.3.

structures is the problem of constructing all their local invariants, which can be thought of as the *natural, intrinsic* scalars of these structures.

In the context of conformal geometry, the first (modern) landmark in understanding *local conformal invariants* was the work of Fefferman and Graham in 1985 [15], where they introduced the *ambient metric*. This allows one to construct local conformal invariants of any order in odd dimensions, and up to order $\frac{n}{2}$ in even dimensions. The question is then whether *all* invariants arise via this construction.

The subsequent work of Bailey, Eastwood and Graham [9] proved that this is indeed true in odd dimensions; in even dimensions, they proved that the result holds when the weight (in absolute value) is bounded by the dimension. The ambient metric construction in even dimensions was recently extended by Graham and Hirachi, [16]; this enabled them to identify in a satisfactory way *all* local conformal invariants, even when the weight (in absolute value) exceeds the dimension.

An alternative construction of local conformal invariants can be obtained via the *tractor calculus* introduced by Bailey, Eastwood and Gover in [8]. This construction bears a strong resemblance to the Cartan conformal connection, and to the work of T.Y. Thomas in 1934, [18]. The tractor calculus has proven to be very universal; tractor bundles have been constructed [10] for an entire class of parabolic geometries. The relation between the conformal tractor calculus and the Fefferman–Graham ambient metric has been elucidated in [11].

The present work [2–7], while pertaining to the question above (given that it ultimately deals with the algebraic form of local *Riemannian* and *conformal* invariants), nonetheless addresses a different *type* of problem: We here consider Riemannian invariants $P(g)$ for which the *integral* $\int_{M^n} P(g)\, dV_g$ remains invariant under conformal changes of the underlying metric; we then seek to understand the possible algebraic form of the *integrand* $P(g)$, ultimately proving that it can be de-composed in the way that Deser and Schwimmer asserted. It is thus not surprising that the prior work on the construction and understanding of local *conformal* invariants plays a central role in this endeavor, in the papers [3,4].

On the other hand, a central element of our proof is the (roughly outlined above) "fundamental Proposition 2.1",[11] which deals *exclusively* with algebraic properties of the *classical* scalar Riemannian invariants.[12] The "fundamental Proposition 2.1" makes no reference to integration; it is purely a statement concerning algebraic properties of *local Riemannian invariants*. While the author was led to the main algebraic propositions in [2,3] out of the strategy that he felt was necessary to solve the Deser–Schwimmer conjecture, they can be thought of as results with an independent interest. The *proof* of these propositions, presented in the present paper and in [6,7] is in fact not particularly intuitive. It is the author's sincere hope that deeper insight (and hopefully a more intuitive proof) will be obtained in the future as to *why* these algebraic propositions hold.

2. The fundamental proposition

In order to state and prove the fundamental proposition we will need to introduce a lot of terminology.

[11] This proposition is a generalization of the main algebraic Propositions 5.1, 3.1, 3.2 in [2,3].

[12] We refer the reader to the introduction of [2] for a detailed discussion of these.

2.1. Definitions and terminology

We will be considering (complete or partial) contractions $C_g^{i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ of length $\sigma+u$ (with no internal contractions) in the form:

$$pcontr\big(\nabla^{(m_1)}R_{ijkl}\otimes\cdots\otimes\nabla^{(m_s)}R_{ijkl}\otimes\nabla^{(b_1)}\Omega_1\otimes\cdots\otimes\nabla^{(b_p)}\Omega_p\otimes\nabla\phi_1\otimes\cdots\otimes\nabla\phi_u\big);\tag{2.1}$$

here $\sigma=s+p$ and ${}_{i_1},\dots,{}_{i_\alpha}$ are the free indices in $C_g^{i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$.

Definition 2.1. Any (complete or partial) contraction in the form (2.1) will be called acceptable if:

1. Each of the free indices belongs to a factor $\nabla^{(m)}R_{ijkl}$ or $\nabla^{(b)}\Omega_h$.
2. Each of the factors $\nabla\phi_h$ contracts against a factor $\nabla^{(m)}R_{ijkl}$ or $\nabla^{(a)}\Omega_f$.
3. Each of the factors $\nabla^{(a)}\Omega_f$ satisfies $a\geqslant 2$.

More generally, we will also be considering tensor fields $C_g^{i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,\phi_{z_1},\dots,\phi_{z_u},\phi'_{z_{u+1}},\dots,\phi'_{z_{u+d}},\tilde{\phi}_{z_{u+d+1}},\dots,\tilde{\phi}_{z_{u+d+y}})$ of length $\sigma+u$ (with no internal contractions) in the form[13]:

$$\begin{aligned}&pcontr\big(\nabla^{(m_1)}R_{ijkl}\otimes\cdots\otimes\nabla^{(m_{\sigma_1})}R_{ijkl}\otimes S_*\nabla^{(\nu_1)}R_{ijkl}\otimes\cdots\otimes S_*\nabla^{(\nu_t)}R_{ijkl}\otimes\\&\quad\nabla^{(b_1)}\Omega_1\otimes\cdots\otimes\nabla^{(b_p)}\Omega_p\otimes\nabla\phi_{z_1}\otimes\cdots\otimes\nabla\phi_{z_w}\otimes\nabla\phi'_{z_{w+1}}\\&\quad\otimes\cdots\otimes\nabla\phi'_{z_{w+d}}\otimes\cdots\otimes\nabla\tilde{\phi}_{z_{w+d+1}}\otimes\cdots\otimes\nabla\tilde{\phi}_{z_{w+d+y}}\big).\end{aligned}\tag{2.2}$$

Here, the functions $\tilde{\phi}_a,\phi'_b$ are the same as the functions ϕ_a,ϕ_b. The symbols $\tilde{}$ and $'$ are used only to illustrate the *kind* of indices that these factors are contracting against (we will explain this in the next definition).

The notion of *acceptability* for contractions in the form (2.2) is a generalization of Definition 2.1:

Definition 2.2. We will call a (complete or partial) contraction in the form (2.2) acceptable if the following conditions hold:

1. $\{z_1,\dots,z_{w+d+y}\}=\{1,\dots,u\}$. Also, each of the free indices must belong to a factor $\nabla^{(m)}R_{ijkl}$, $\nabla^{(b)}\Omega_h$ or $S_*\nabla^{(\nu)}R_{ijkl}$. In addition, the factors $\nabla^{(b)}\Omega_f$ must have $b\geqslant 2$.
2. The factors $\nabla\phi_h,\nabla\phi'_h,\nabla\tilde{\phi}_h$ contract according to the following pattern: Each of the factors $\nabla\phi_h$ contracts against a derivative index in a factor $\nabla^{(m)}R_{ijkl}$ or $\nabla^{(b)}\Omega_f$. Each of the factors $\tilde{\phi}_h$ contracts against the index ${}_i$ of some factor $S_*\nabla^{(\nu)}R_{ijkl}$. Conversely, each index ${}_i$ in any factor $S_*\nabla^{(\nu)}R_{ijkl}$ contracts against some factor $\nabla\tilde{\phi}_h$. Lastly, each factor $\nabla\phi'_h$ contracts against some factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$, but necessarily against one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$.

[13] We recall that $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ stands for the symmetrization of the tensor $\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ over the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$.

Definition 2.3. For any (complete or partial) contraction in the form (2.1) or (2.2), we define its *real length* to be the number of its factors *if we exclude* the factors $\nabla\phi_h$, $\nabla\tilde{\phi}_h$, $\nabla\phi'_h$. (So for contractions in the form (2.2) the real length is $\sigma_1 + t + p$.)

We now introduce the notions of *weak*, *simple*, *double* and *refined-double* characters for acceptable contractions $C^{i_1\dots i_a}_g$ in the form (2.2). Since these definitions are rather technical, we first give the gist of these notions: The weak character encodes the pattern of *which* factors in $C^{i_1\dots i_a}_g$ the various terms $\nabla\phi_h$, $h = 1, \dots, u$ are contracting against. The simple character encodes the above, but also encodes whether each given factor $\nabla\phi_h$ that contracts against a factor $T = S_*\nabla^{(\nu)}R_{ijkl}$ is contracting against the index ${}_i$, or one of the indices ${}_{r_1}, \dots, {}_{r_\nu}, {}_j$. The double character encodes the simple character, but also encodes how the free indices are distributed among the different factors in $C^{i_1\dots i_a}_g$ (i.e. how many free indices belong to each factor). Finally, the refined double character encodes the same information as the double character, but also encodes whether the free indices are *special indices*.[14]

Now we present the proper definitions of the different notions of "character".

Definition 2.4. Consider any acceptable complete or partial contraction in either of the forms (2.1) or (2.2). The weak character $\vec{\kappa}_{weak}$ is defined to be a pair of two lists of sets: (L_1, L_2). L_1 stands for the list $(S_1, \dots, S_p)$ where each S_t stands for the set of numbers r for which $\nabla\phi_r$ contracts against the factor $\nabla^{(b_t)}\Omega_t$. L_2 stands for the list of sets $(S_1, \dots, S_{\sigma-p})$ where each S_t stands for the set of numbers r for which $\nabla\phi_r$ contracts against the tth curvature factor in (2.1) or (2.2) (in the latter case the curvature factor may be in the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$).

Definition 2.5. Consider complete or partial contractions in the form (2.2); we define the simple character $\vec{\kappa}_{simp}$ to be a triplet of lists: (L_1, L_2, L_3).

L_1 is as above. L_2 stands for the list of sets $(S_1, \dots, S_{\sigma_1})$ where each S_t stands for the set of labels r of the factors $\nabla\phi_r$ that contract against the tth factor $\nabla^{(m_t)}R_{ijkl}$ in the first line of (2.2). L_3 is a sequence of pairs of sets: $L_3 = [(\{\alpha_1\}, S_1), \dots, (\{\alpha_t\}, S_t)]$, where α_w stands for the label of the factor $\nabla\tilde{\phi}_{a_w}$ against which the index ${}_i$ in the wth factor in the second line of (2.2) is contracting. S_w stands for the set of labels r of the factors $\nabla\phi'_r$ against which the wth factor is contracting in (2.2).

Now, we define the double character. We note that this notion is defined for tensor fields in the form (2.2) that *do not* have both indices ${}_i, {}_j$ or ${}_k, {}_l$ in a factor $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$ being free.

Definition 2.6. Consider complete or partial contractions in the form (2.2); we define the double character to be the union of two triplets of lists: $\vec{\kappa}_{doub} = (L_1, L_2, L_3)|(H_1, H_2, H_3)$. Here L_1, L_2, L_3 are as above. H_1 stands for the list $(h_1, \dots, h_p)$, where h_t stands for the number of free indices that belong to the factor $\nabla^{(b_t)}\Omega_t$. H_2 stands for the list of numbers $(h_1, \dots, h_{\sigma_1})$, where each h_i stands for the number of free indices that belong to the ith factor in the first line of (2.2). H_3 stands for the set of numbers $(h_1, \dots, h_t)$ where h_u stands for the number of free indices that belong to the uth factor on the second line of (2.2).

[14] Meaning that they are internal indices in some $\nabla^{(m)}R_{ijkl}$ or indices ${}_k, {}_l$ in some $S_*\nabla^{(\nu)}R_{ijkl}$.

We now define the *refined* double character of tensor fields in the form (2.2). For that purpose, we will be paying special attention to the free indices ${}_{i_f}$ that are internal indices in some factor $\nabla^{(m)} R_{ijkl}$ or are one of the indices ${}_{k}, {}_{l}$ in one of the factors $S_*\nabla^{(\nu)} R_{ijkl}$. We will be calling those free indices *special free indices*.

Definition 2.7. Consider complete or partial contractions in the form (2.2); we define its *refined double character* to be the union of two triplets of sets: $\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}} = (L_1, L_2, L_3)|(H_1, \tilde{H}_2, \tilde{H}_3)$ where the sets L_1, L_2, L_3, H_1 are as before, whereas:

$\tilde{H}_2$ stands for the list of sets $(\tilde{h}_1, \dots, \tilde{h}_{\sigma_1})$ where $\tilde{h}_k$ stands for the following: If the kth factor $\nabla^{(m)} R_{ijkl}$ has no special free indices then $\tilde{h}_k = h_k$ (same as for the double character). If it contains one special free index then $\tilde{h}_k = \{h_k\} \cup \{*\}$. Finally, if it contains two special free indices then $\tilde{h}_k = \{h_k\} \cup \{**\}$.

$\tilde{H}_3$ stands for the list of sets $(\tilde{h}_1, \dots, \tilde{h}_{\sigma_2})$ where $\tilde{h}_k$ stands for the following: If the kth factor $S_*\nabla^{(\nu)} R_{ijkl}$ has no special free indices then $\tilde{h}_k = h_k$ (same as for the double character). If it contains one special free index then $\tilde{h}_k = \{h_k\} \cup \{*\}$.

The elements $\{**\}$, $\{*\}$ above will be called marks.

Now, we will introduce a notion of *equivalence* for the characters (weak, simple, double or refined double) of tensor fields.

Definition 2.8. We say that two (complete or partial) contractions in the form (2.2) have equivalent (simple, double or refined double) characters if their (simple, double or refined double) characters can be made equal by permuting factors among the first two lines of (2.2).

More generally, we will say that two (complete or partial) contractions in the more general form (2.1) (possibly of different rank) have equivalent weak characters if we can permute their curvature factors and make their weak characters equal.

We thus see that the various "characters" we have defined can be thought of as abstract lists, which are equipped with a natural notion of equivalence. We note that we will occasionally be speaking of a (weak, simple, double or refined double) character abstractly, without specifying a (complete or partial) contraction or tensor field that it represents. Furthermore, we have seen that the notions of weak, simple, double and refined double characters are *graded*, in the sense that each of these four notions contains all the information of the previous ones. Now, given a simple character $\vec{\kappa}_{simp}$ we define $\mathit{Weak}(\vec{\kappa}_{simp})$ to stand for the weak character that corresponds to that simple character. Analogously, given any refined double character $\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}}$, we let $\mathit{Simp}(\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}})$ stand for the simple character that corresponds to that refined double character and also $\mathit{Weak}(\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}})$ to stand for the weak character that corresponds to that refined double character.

Now, we will introduce a weak notion of *ordering* for simple and refined double characters. This notion is "weak" in the sense that we will be specifying a particular simple or refined double character $\vec{\kappa}_{simp}$ and $\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}}$ respectively, and we will define what it means for complete or partial contractions (in the form (2.1) or (2.2)) to be *subsequent* to $\vec{\kappa}_{simp}$ and $\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}}$ respectively. This relation is *not* transitive.

Definition 2.9. Given any contraction in the form (2.2), we consider any simple character $\vec{\kappa}_{simp}$ or refined double character $\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}}$ and we let the defining set $\mathit{Def}(\vec{\kappa}_{simp})$, $\mathit{Def}(\vec{\kappa}_{\mathit{ref}\text{-}\mathit{doub}})$ to be the set of labels r for which $\nabla\tilde{\phi}_r$ is contracting against an internal index ${}_i$ in some factor $S_*\nabla^{(\nu)} R_{ijkl}$.

We now consider any general complete or partial contraction $C_g^{i_1 \dots i_a}$ in the form (2.1) or (2.2), with a weak character $Weak(\vec{\kappa}_{simp})$ or $Weak(\vec{\kappa}_{ref\text{-}doub})$ respectively. We say that C_g or $C_g^{i_1 \dots i_a}$ is simply subsequent to $\vec{\kappa}_{simp}$ or $\vec{\kappa}_{ref\text{-}doub}$ if for at least one number $\nu \in Def(\vec{\kappa}_{simp})$ (or $\nu \in Def(\vec{\kappa}_{ref\text{-}doub})$), the factor $\nabla\tilde{\phi}_\nu$ in C_g or $C_g^{i_1 \dots i_a}$ is contracting against a derivative index. This terminology extends to linear combinations.

Now, we will introduce a partial ordering among refined double characters $\vec{\kappa}_h$ with $Simp(\vec{\kappa}_h) = \vec{\kappa}_{simp}$, where $\vec{\kappa}_{simp}$ is a *fixed* simple character. To do this, some more notation is needed:

For a given tensor field in the form (2.2), with a refined double character $\vec{\kappa}_{ref\text{-}doub}$, we define $Def^*(\vec{\kappa}_{ref\text{-}doub})$ to stand for the subset of $Def(\vec{\kappa}_{ref\text{-}doub})$ which consists of those labels a_w for which $\nabla\tilde{\phi}_{a_w}$ contracts against a factor $S_*\nabla^{(\nu)}R_{ijkl}$ where one of the indices $_k$ or $_l$ is a free index.

Definition 2.10. We compare two refined double characters $\vec{\kappa}_1, \vec{\kappa}_2$ with $Simp(\vec{\kappa}_1) = Simp(\vec{\kappa}_2)$ according to their $*$-decreasing rearrangements:

By a $*$-decreasing rearrangement of any refined double character, we mean the rearrangement of the lists $\tilde{H}_2, \tilde{H}_3$ (see Definition 2.7 above) so that the elements in $\tilde{H}_2$ with a mark $\{**\}$ must come first (and the elements with such factors are arranged in decreasing rearrangement). Then among the rest of the elements, the ones with a mark $\{*\}$ must come first (and the elements with such a mark are arranged in decreasing rearrangement). Then, the elements without a mark will come in the end, arranged in decreasing rearrangement. Furthermore, for the lists $\tilde{H}_3$ $*$-decreasing rearrangement means that the elements in $\tilde{H}_3$ corresponding to a factor $S_*\nabla^{(\nu)}R_{ijkl}$ *which is contracting against some factor* $\nabla\phi'_h$ *in* $\vec{\kappa}_{simp}$ will come first in the list (and they are arranged in decreasing rearrangement), and then come the elements corresponding to a factor $S_*\nabla^{(\nu)}R_{ijkl}$ *which are not contracting against any factor* $\nabla\phi'_h$ *in* $\vec{\kappa}_{simp}$, and those are also arranged in decreasing rearrangement.

Now, to compare the refined double characters $\vec{\kappa}_1, \vec{\kappa}_2$ according to their $*$-decreasing rearrangements means that we take their lists $\tilde{H}_3$ in $*$-decreasing rearrangement (see above) and order them lexicographically according to these rearranged lists. If they are still equivalent, we take the lists $\tilde{H}_2$ in $*$-decreasing rearrangement and order them lexicographically. If they are still equivalent, we take the decreasing rearrangement of the lists H_1 and compare them lexicographically. "Lexicographically" here means that we compare the first elements in the $*$-rearranged lists, then the second etc. The list with the first larger element is precedent (the converse of "subsequent") to the other list.

If $\vec{\kappa}_1, \vec{\kappa}_2$ are still equivalent after all these comparisons, we say that the refined double characters $\vec{\kappa}_1, \vec{\kappa}_2$ are "equipolent". We remark that two refined double characters $\vec{\kappa}_1, \vec{\kappa}_2$ can be equipolent without being equivalent.

A final note before stating our proposition. We remark that the simple and the weak characters are independent of the *rank* of the tensor field. On the other hand, the double and the refined double characters *do* depend on the rank of the tensor fields: Two double characters or two refined double characters cannot be equivalent if the tensor fields do not have the same rank. We will then extend the notion of double character and refined double character as follows: We consider any tensor field $C^{i_1 \dots i_\beta}$ in the form (2.2), and also any number $\alpha \leqslant \beta$. We then define the α-double character or the α-refined double character of $C^{i_1 \dots i_\beta}$ in the same way as for Definitions 2.5 and 2.6, with the extra restriction that whenever we refer to a free index $_{i_d}$, we will mean that $d \leqslant \alpha$.

We notice that with this new definition, we can have two tensor fields $C^{i_1\dots i_\beta}$, $C^{i_1\dots i_\alpha}$ with $\alpha<\beta$, so that the double character of $C^{i_1\dots i_\alpha}$ and the α-double character of $C^{i_1\dots i_\beta}$ are equivalent. The same is true of refined double characters. We note that this notion depends on the order of the free indices in $C^{i_1\dots i_\beta}$.

Furthermore, we note that we will sometimes be referring to a u-simple character $\vec{\kappa}_{simp}$ to stress that the information encoded will refer to the u factors $\nabla\phi_1,\dots,\nabla\phi_u$. Analogously, we will sometimes refer to a (u,μ)-refined double character to stress that the information encoded refers to the u factors $\nabla\phi_1,\dots,\nabla\phi_u$ and the μ free indices ${}_{i_1},\dots,{}_{i_\mu}$.

Forbidden cases: Now, we introduce a last definition of certain "forbidden cases" in which the Proposition 2.1 will not apply. Firstly we introduce a definition.

Definition 2.11. Given a simple character $\vec{\kappa}_{simp}$ and any factor $T=S_*\nabla^{(\nu)}R_{ijkl}$ in $\vec{\kappa}_{simp}$, we will say that T is simple if it is not contracting against any factors $\nabla\phi'_h$ in $\vec{\kappa}_{simp}$.

Also, given a factor $T=\nabla^{(B)}\Omega_k$, we will say that T is simple if it is not contracting against any factor $\nabla\phi_h$ in $\vec{\kappa}_{simp}$.

We recall that σ_2 stands for the number of factors $S_*\nabla^{(\nu)}R_{ijkl}$ in $\vec{\kappa}_{simp}$.

Definition 2.12. A tensor field in the form (2.2) will be called "forbidden" only when $\sigma_2>0$, under the following additional restrictions:

If $\sigma_2=1$, it will be forbidden if:

1. Any factor $\nabla^{(m)}R_{ijkl}$ must have all its derivative indices contracting against factors $\nabla\phi_x$ and contain no free indices.
2. Any factor $\nabla^{(p)}\Omega_h$ must have $p=2$, be simple, and contain no free indices.
3. The factor $S_*\nabla^{(\nu)}R_{ijkl}$ must have $\nu=0$, be simple, and contain exactly one (special) free index.

If $\sigma_2>1$, it will be forbidden if:

1. Any factor $\nabla^{(m)}R_{ijkl}$ must have all its derivative indices contracting against factors $\nabla\phi_x$ and contain at most one (necessarily special) free index.
2. Any factor $\nabla^{(p)}\Omega_h$ must have $p=2$. If it is simple, it can contain at most one free index; if it is non-simple, then it must contract against exactly one factor $\nabla\phi_h$ and contain no free indices.
3. Any factors $S_*\nabla^{(\nu)}R_{ijkl}$ must have $\nu=0$, be simple, and contain at most one free index. Moreover at least one of the factors S_*R_{ijkl} must contain a special free index.

Finally, we note that in stating Proposition 2.1 we will be formally considering linear combinations of tensor fields of different ranks.

Proposition 2.1. *Consider two linear combinations of acceptable tensor fields in the form* (2.2)*:*

$$\sum_{l\in L_\mu} a_l C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

$$\sum_{l\in L_{>\mu}} a_l C_g^{l,i_1\dots i_{\beta_l}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

where each tensor field above has real length $\sigma \geqslant 3$ and a given simple character $\vec{\kappa}_{simp}$. We assume that for each $l\in L_{>\mu}$, $\beta_l \geqslant \mu+1$. We also assume that none of the tensor fields of maximal refined double character in L_μ are "forbidden" (see Definition 2.12*).*

We denote by

$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

a generic linear combination of complete contractions (not necessarily acceptable) in the form (2.1) *that are simply subsequent to $\vec{\kappa}_{simp}$.*[15] *We assume that:*

$$\begin{aligned}&\sum_{l\in L_\mu} a_l X\,div_{i_1}\dots X\,div_{i_\mu} C_g^{l,i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&\quad+\sum_{l\in L_{>\mu}} a_l X\,div_{i_1}\dots X\,div_{i_{\beta_l}} C_g^{l,i_1\dots i_{\beta_l}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&\quad+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0.\end{aligned}\tag{2.3}$$

We draw our conclusion with a little more notation: We break the index set L_μ into subsets $L^z, z\in Z$, (Z is finite) with the rule that each L^z indexes tensor fields with the same refined double character, and conversely two tensor fields with the same refined double character must be indexed in the same L^z. For each index set L^z, we denote the refined double character in question by $\vec{L}^z$. Consider the subsets L^z that index the tensor fields of *maximal* refined double character.[16] We assume that the index set of those z's is $Z_{Max}\subset Z$.

We claim that for each $z\in Z_{Max}$ there is some linear combination of acceptable $(\mu+1)$-tensor fields,

$$\sum_{r\in R^z} a_r C_g^{r,i_1\dots i_{\alpha+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

where each $C_g^{r,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ has a μ-double character $\vec{L}_1^z$ and also the same set of factors $S_*\nabla^{(\nu)}R_{ijkl}$ as in $\vec{L}^z$ contain special free indices, so that:

$$\begin{aligned}&\sum_{l\in L^z} a_l C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\\&\quad-\sum_{r\in R^z} a_r X\,div_{i_{\mu+1}} C_g^{r,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\end{aligned}$$

[15] Of course if $Def(\vec{\kappa}_{simp})=\emptyset$ then by definition $\sum_{j\in J}\dots=0$.

[16] Note that in any set S of μ-refined double characters with the same simple character there exists a subset S' consisting of the maximal refined double characters.

$$= \sum_{t \in T_1} a_t C_g^{t,i_1 \dots i_\mu}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \upsilon \dots \nabla_{i_\mu} \upsilon, \tag{2.4}$$

modulo complete contractions of length $\geqslant \sigma + u + \mu + 1$. Here each

$$C_g^{t,i_1 \dots i_\mu}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$$

is acceptable and is either simply or doubly subsequent to $\vec{L}^z$.[17]

Trivial observation: We recall that when a tensor field is acceptable, then by definition it does not have two free indices (say $_{i_q}, _{i_w}$) that are indices $_i, _j$ or $_k, _l$ in the same curvature factor. Thus, such tensor fields are not allowed in our proposition hypothesis, (2.3). Nonetheless, the conclusion of the above proposition would still be true if we did allow such tensor fields: It suffices to observe that this sublinear combination would vanish both in the hypothesis of our proposition and in its conclusion. This is straightforward, by virtue of the antisymmetry of those indices.

Now, Proposition 2.1 has a corollary which will be used more often than the proposition itself:

Corollary 1. *Assume Eq.* (2.3) *(and again assume that the maximal refined double characters appearing there are not "forbidden"). We then claim that there is a linear combination of acceptable* $(\mu + 1)$*-tensor fields*

$$\sum_{h \in H} a_h C_g^{h,i_1,\dots,i_{\mu+1}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$$

with simple character $\vec{\kappa}_{simp}$*, so that:*

$$\sum_{l \in L_\mu} a_l C_g^{l,i_1 \dots i_\mu}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \upsilon \dots \nabla_{i_\mu} \upsilon$$
$$+ \sum_{h \in H} a_h X\, div_{i_{\mu+1}} C_g^{h,i_1 \dots i_{\mu+1}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \upsilon \dots \nabla_{i_\mu} \upsilon$$
$$= \sum_{t \in T} a_t C_g^{t,i_1 \dots i_\mu}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \upsilon \dots \nabla_{i_\mu} \upsilon, \tag{2.5}$$

modulo complete contractions of length $\geqslant \sigma + u + \mu + 1$*. Here the right-hand side stands for a generic linear combination of acceptable tensor fields that are simply subsequent to* $\vec{\kappa}_{simp}$*.*

Proof that Corollary 1 follows from Proposition 2.1. We will prove our claim by an induction.

We consider all the (u, μ)-double characters $\vec{\kappa}$ with the property that $Simp(\vec{\kappa}) = \vec{\kappa}_{simp}$. It follows by definition that there is a finite number of such refined double characters, so we denote their set by $\{Doub_1(\vec{L_\mu}), \dots, Doub_U(\vec{L_\mu})\}$. We view the above as an ordered set, with the additional restriction that for each a, b, $1 \leqslant a < b \leqslant U$, $Doub_a(\vec{L_\mu})$ is not doubly subsequent to $Doub_b(\vec{L_\mu})$. Accordingly, we break the index set L_μ into subsets $L^1, \dots, L^U$ (where if $l \in L^t$ then $C_g^{l,i_1 \dots i_\mu}$ has a refined double character $Doub_t(\vec{L_\mu})$).

[17] Recall that "simply subsequent" means that the simple character of $C_g^{t,i_1 \dots i_\mu}$ is subsequent to $Simp(\vec{L}^z)$.

We then claim the following statement: We inductively assume that for some f, $1 \leqslant f \leqslant U$, we have shown that there is a linear combination of acceptable $(\mu+1)$-tensor fields with simple characters $\vec{\kappa}_{simp}$, say

$$\sum_{h \in H''} a_h C_g^{l,i_1 \ldots i_{\mu+1}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),$$

so that:

$$\begin{aligned}
&\sum_{w \leqslant f} \sum_{l \in L^w} a_l C_g^{l,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \upsilon \ldots \nabla_{i_\mu} \upsilon \\
&\quad - \sum_{h \in H''} a_h X\, div_{i_{\mu+1}} C_g^{l,i_1 \ldots i_{\mu+1}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \upsilon \ldots \nabla_{i_\mu} \upsilon \\
&= \sum_{w=f+1}^{U} \sum_{d \in D^w} a_d C_g^{d,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \upsilon \ldots \nabla_{i_\mu} \upsilon \\
&\quad + \sum_{t \in T} a_t C_g^{t,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \upsilon \ldots \nabla_{i_\mu} \upsilon,
\end{aligned} \tag{2.6}$$

where each $C^{d,i_1 \ldots i_\mu}$, $d \in D^w$ has a refined double character $Doub_w(\vec{L})$. Write:

$$\begin{aligned}
&\sum_{w=f+1}^{U} \sum_{l \in L^w} a_l C_g^{l,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \\
&\quad + \sum_{w=f+1}^{U} \sum_{d \in D^w} a_d C_g^{d,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \\
&= \sum_{w=f+1}^{U} \sum_{y \in Y^w} a_y C_g^{y,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),
\end{aligned} \tag{2.7}$$

where the index sets Y^w stand for the index sets that arise when we group up all the acceptable μ-tensor fields of the same refined double character. We then claim that for $w = f+1$ there is a linear combination of acceptable $(\mu+1)$-tensor fields, say

$$\sum_{h \in H'''} a_h C_g^{l,i_1 \ldots i_{\mu+1}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),$$

so that:

$$\begin{aligned}
&\sum_{y \in Y^{f+1}} a_y C_g^{y,i_1 \ldots i_\mu}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \upsilon \ldots \nabla_{i_\mu} \upsilon \\
&\quad - \sum_{h \in H'''} a_h X\, div_{i_{\mu+1}} C_g^{l,i_1 \ldots i_{\mu+1}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \upsilon \ldots \nabla_{i_\mu} \upsilon
\end{aligned}$$

$$= \sum_{t\in T} a_t C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$+ \sum_{k>f+1}\sum_{y\in Y^k} a_y C_g^{y,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon. \tag{2.8}$$

It is clear that if we can show the above, then since the set $\{Doub_1(\vec{L_\mu}),\dots,Doub_U(\vec{L_\mu})\}$ is finite, we will have shown our corollary.

But (2.8) is not difficult to show: Since (2.6) holds formally we can replace the $\nabla\upsilon$s by $X\,div$s (see the last lemma in the Appendix of [2]) and substitute into (2.3) to obtain:

$$\sum_{w=f+1}^{U}\sum_{y\in Y^w} a_y X\,div_{i_1}\dots X\,div_{i_\mu} C_g^{y,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$+ \sum_{h\in H} a_h X\,div_{i_1}\dots X\,div_{i_\mu} X\,div_{i_{\mu+1}} C_g^{l,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$+ \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) = 0. \tag{2.9}$$

Because the sum in the first line of (2.9) starts at $w=f+1$, it follows that one of the maximal sublinear combinations in the first line of (2.9) is the sublinear combination

$$\sum_{y\in Y^{f+1}} a_y C_g^{y,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u).$$

Therefore, (2.8) follows immediately from the conclusion of Proposition 2.1. □

2.2. The main algebraic propositions in [2,3] follow from Corollary 1

We will now show how Proposition 5.1 in [2] and Propositions 3.1, 3.2 in [3] follow from Corollary 2.1. For the first two, this is immediate: Observe that in case of Proposition 5.2 in [2], the simple character of the tensor fields in Eq. (2.3) just encodes the fact that there are σ_1 factors $\nabla^{(m)}R_{ijkl}$ and p factors $\nabla^{(y)}\Omega_h$, $h=1,\dots,p$; in the setting of Proposition 3.1 in [3] it additionally encodes the fact that the tensor fields also contain a factor $\nabla\phi$ $(=\nabla\phi_1)$ which either contracts against a factor $\nabla^{(y)}\Omega$ or against a derivative index of a factor $\nabla^{(m)}R_{ijkl}$, for each of the tensor fields in the hypothesis of that proposition. There are no factors $S_*\nabla^{(\nu)}R_{ijkl}$ in this setting, thus $\sum_{j\in J} a_j\dots = 0$, both in the hypothesis and in the conclusion of Corollary 1. Thus, the claims of these two propositions follow directly from the conclusion of Corollary 1 by just replacing the expression $\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$ by a symmetrization over the indices ${}^{i_1},\dots,{}^{i_\mu}$.[18]

On the other hand, in order to derive Proposition 3.2 in [3] from Corollary 1, we have to massage the hypothesis of that proposition in order make it fit with the hypothesis of Corollary 1. For each tensor field $C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi)$ and $C_g^{l,i_1\dots i_{\beta_l}}(\Omega_1,\dots,\Omega_p,\phi)$ in Proposition 3.2 in [3], we denote by $\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi)$ and $\tilde{C}_g^{l,i_1\dots i_{\beta_l}}(\Omega_1,\dots,\Omega_p,\phi)$ the tensor fields that

[18] See the remark after the statement of Proposition 5.1 in [2].

arise from them by formally replacing the expression $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi$ by $S_*\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi$. We then observe (by virtue of the second Bianchi identity) that:

$$C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi)=\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi)+\sum_{j\in J}a_jC^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi), \tag{2.10}$$

$$C^{l,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi)=\tilde{C}^{l,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi)+\sum_{j\in J}a_jC^{j,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi). \tag{2.11}$$

Notice that the tensor fields $\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi)$, $\tilde{C}^{l,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi)$ are all in the form (2.2) and they all have the same *simple character*, which we denote by $\vec{\kappa}_{simp}$. The complete contractions in $\sum_{j\in J}a_j\dots$ in the hypothesis of Proposition 3.2 in [3] *and* the complete contractions in $X\,div_{i_1}\dots X\,div_{i_\mu}C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi)$, $X\,div_{i_1}\dots X\,div_{i_{\beta_l}}C^{j,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi)$ are all *simply subsequent* to $\vec{\kappa}_{simp}$.

Thus, replacing the above into the hypothesis of Proposition 3.2 in [3] we obtain an equation to which Corollary 1 can be applied.[19] We derive that there is a linear combination of acceptable tensor fields, $\sum_{h\in H}a_hC^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi)$ in the form (2.2), each with a simple character $\vec{\kappa}_{simp}$ so that:

$$\begin{aligned}\sum_{l\in L_1}a_l\tilde{C}^{l,(i_1\dots i_\mu)}_g(\Omega_1,\dots,\Omega_p,\phi)-X\,div_{i_{\mu+1}}\sum_{h\in H}a_hC^{h,(i_1\dots i_\mu)i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi)\\ =\sum_{j\in J}a_jC^{J,(i_1\dots i_\mu)}_g(\Omega_1,\dots,\Omega_p,\phi).\end{aligned} \tag{2.12}$$

Combined with Eqs. (2.10), (2.11) above, (2.12) is precisely our desired conclusion.

3. Proof of Proposition 2.1: set up an induction and reduce to Lemmas 3.1, 3.2, 3.5 below

3.1. The proof of Proposition 2.1 via an induction

We will prove Proposition 2.1 by a multiple induction on different parameters (see the enumeration below). We re-write the hypothesis of Proposition 2.1, for reference purposes.

We are given two linear combinations of acceptable tensor fields in the form (2.2) all with a given simple character $\vec{\kappa}_{simp}$. We have the linear combination:

$$\sum_{l\in L_\mu}a_lC^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

[19] Notice that the extra requirement in Proposition 3.2 ensures that we do not fall under “a forbidden case” of Corollary 1.

for which all the tensor fields have rank μ, and also the linear combination:

$$\sum_{l\in L\setminus L_\mu} a_l C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

for which all the tensor fields have rank strictly greater than μ (we should denote the rank by $a_l > \mu$ instead of a, to stress that the tensor fields in $L\setminus L_\mu$ have different ranks—however we will write $C^{l,i_1\dots i_a}_g$, thus abusing notation). We are assuming an equation:

$$\begin{aligned}\sum_{l\in L_\mu} a_l X\,div_{i_1}\dots X\,div_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\ +\sum_{l\in L\setminus L_\mu} a_l X\,div_{i_1}\dots X\,div_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\ +\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0,\end{aligned} \tag{3.1}$$

which holds modulo complete contractions of length $\geqslant \sigma+u+1$.[20] (Recall that σ stands for the number of factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$, $\nabla^{(B)}\Omega_x$ in $\vec{\kappa}_{simp}$.) We recall that each C^j is simply subsequent to $\vec{\kappa}_{simp}$.

The inductive assumptions: We explain the inductive assumptions on Proposition 2.1 in detail:

Denote the left-hand side of Eq. (3.1) by $L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ or just L_g for short. For the complete contractions in L_g, σ_1 will stand for the number of factors $\nabla^{(m)}R_{ijkl}$ and σ_2 will stand for the number of factors $S_*\nabla^{(\nu)}R_{ijkl}$. Also Φ will stand for the total number of factors $\nabla\phi, \nabla\tilde{\phi}, \nabla\phi'$ and $-n$ will stand for the weight of the complete contractions involved.

1. We assume that Proposition 2.1 is true for all linear combinations $L_{g^{n'}}$ with weight $-n'$, $n'<n$, n' even, that satisfy the hypotheses of our proposition.
2. We assume that Proposition 2.1 is true for all linear combinations L_g of weight $-n$ and real length $\sigma'<\sigma$, that satisfy the hypotheses of our proposition.
3. We assume that Proposition 2.1 is true for all linear combinations L_g of weight $-n$ and real length σ, with $\Phi'>\Phi$ factors $\nabla\phi, \nabla\tilde{\phi}, \nabla\phi'$, that satisfy the hypotheses of our proposition.
4. We assume that Proposition 2.1 is true for all linear combinations L_g of weight $-n$ and real length σ, Φ factors $\nabla\phi, \nabla\tilde{\phi}, \nabla\phi'$ and with *fewer than* $\sigma_1+\sigma_2$ *curvature factors* $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$, provided L_g satisfies the hypotheses of our proposition.

We will then show Proposition 2.1 for the linear combinations L_g with weight $-n$, real length σ, Φ factors $\nabla\phi$, $\nabla\phi'$, $\nabla\tilde{\phi}$ and with $\sigma_1+\sigma_2$ curvature factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$. So we are proving our proposition by a multiple induction on the parameters n, σ, Φ, $\sigma_1+\sigma_2$ of the linear combination L_g. A trivial observation: For each weight $-n$, there are obvious (or assumed) bounds on the numbers σ ($\geqslant 3$), $\sigma-\sigma_1-\sigma_2$ ($\geqslant 0, <n$) and on the number Φ ($\leqslant \frac{n}{2}$). In view of this, we see that if we can show this inductive statement, then Proposition 2.1 will follow by induction.

[20] We have now set $L_\mu\cup L_{>\mu}=L$.

The rest of this series of papers is devoted to proving this inductive step of Proposition 2.1. However, for simplicity we will still refer to "proving Proposition 2.1" rather than "proving the inductive step of Proposition 2.1".

3.2. Reduction of Proposition 2.1 to three lemmas

We will claim three lemmas below: Lemmas 3.1, 3.2 and 3.5.[21] We will then prove in the next section that Proposition 2.1 follows from these three lemmas (apart from some exceptional cases, where we will derive Proposition 2.1 directly—they will be presented in the paper [6] in this series). As these lemmas are rather technical, we give here the gist of their claims, and also indicate, very roughly, how they will imply Proposition 2.1, by virtue of our inductive assumptions above. A rigorous proof of how Lemmas 3.1, 3.2 and 3.5 imply Proposition 2.1 will be given in Section 4 of the present paper.

General discussion of ideas: We distinguish three cases regarding the tensor fields of rank μ appearing in (3.1). In the first case, some of the μ-tensor fields (indexed in L_μ) have special free indices belonging to factors $S_*\nabla^{(\nu)}R_{ijkl}$.[22] In the second case, none of the μ-tensor fields (indexed in L_μ) have special free indices belonging to factors $S_*\nabla^{(\nu)}R_{ijkl}$, but some have special free indices in factors $\nabla^{(m)}R_{ijkl}$.[23] In the third case, there are no special free indices in any μ-tensor field in (3.1). The three Lemmas 3.1, 3.2 and 3.5 correspond to these three cases.

A note: It follows that in the first case above, the μ-tensor fields in (3.1) of *maximal* refined double character will have a special free index in some factor $S_*\nabla^{(\nu)}R_{ijkl}$ (this follows from the definition of *maximal refined double character*, Definition 2.7). It also follows that in the second case the μ-tensor fields of *maximal* refined double character will have a special free index in some factor $\nabla^{(m)}R_{ijkl}$; in the third case, the maximal μ-tensor fields will have no special free indices. We now outline the statements of Lemmas 3.1, 3.2, 3.5:

In the roughest terms, in each of the three cases above, the corresponding lemma states the following: We "canonically" pick out some sub-linear combination of the maximal μ-tensor fields (for this discussion we denote the index set of this sublinear combination by $\overline{L}^{Max}_\mu \subset L_\mu$). In the first two cases, we consider each $C^{l,i_1\dots i_\mu}_g$, $l\in \overline{L}^{Max}_\mu$ and canonically pick out one (or a set of) special free indices. For the purposes of this discussion, we will assume that we have canonically picked out one free index, and we will assume it is the index ${}_{i_1}$ in each $C^{l,i_1\dots i_\mu}_g$, $l\in \overline{L}^{Max}_\mu$.

A rough description of the claim of Lemmas 3.1 and 3.2: For Lemmas 3.1 and 3.2, our claim is roughly an equation of the form:

$$\sum_{l\in \overline{L}^{Max}_\mu} a_l X\,div_{i_2}\dots X\,div_{i_\mu} C^{l,i_1\dots i_\mu}_g \nabla_{i_1}\phi_{u+1}$$
$$+\sum_{\nu\in N} a_\nu X\,div_{i_2}\dots X\,div_{i_\mu} C^{\nu,i_1\dots i_\mu}_g \nabla_{i_1}\phi_{u+1}$$
$$+\sum_{p\in P} a_p X\,div_{i_2}\dots X\,div_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g \nabla_{i_1}\phi_{u+1}+\sum_{j\in J'} a_j C^{j,i_1}_g\nabla_{i_1}\phi_{u+1}=0, \tag{3.2}$$

[21] Lemma 3.5 also relies on certain preparatory Lemmas 3.3, 3.4 which will be proven in [7].

[22] Recall that a special free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is one of the indices ${}_{k},{}_{l}$.

[23] Recall that a special free index that belongs to $\nabla^{(m)}R_{ijkl}$ is one of the indices ${}_{i},{}_{j},{}_{k},{}_{l}$.

which holds modulo longer complete contractions. Here the tensor fields indexed in $\overline{L}^{Max}_{\mu} \cup N \cup P$ are "acceptable" in a suitable sense. The $(\mu - 1)$-tensor fields indexed in $\overline{L}^{Max}_{\mu}$ have a specified "simple character" $\vec{\kappa}'_{simp}$ (in a suitable sense), where this "'simple character" encodes the pattern of which factors the different terms $\nabla\phi_1, \ldots, \nabla\phi_u, \nabla\phi_{u+1}$ are contracting against. Also, all the $(\mu - 1)$-tensor fields indexed in N have this "simple character" $\vec{\kappa}'_{simp}$, but they are "doubly subsequent" (in a suitable sense) to the tensor fields indexed in $\overline{L}^{Max}_{\mu}$. Finally, the complete contractions indexed in J' are "simply subsequent" (in a suitable sense) to $\vec{\kappa}'_{simp}$. *Note:* The tensor fields in (3.2) will not always be in the form (2.2), thus our usual definitions of "character", "subsequent" etc. do not immediately apply.

A rough description of the claim of Lemma 3.5: Now, we can roughly describe the claim of Lemma 3.5 (which is the hardest of the three): We "canonically" pick out a sub-linear combination of the maximal μ-tensor fields in (3.1) (denote the index set of this sublinear combination by $\overline{L}^{Max}_{\mu}$). For each μ-tensor field $C^{l,i_1\ldots i_\mu}_g$, $l \in \overline{L}^{Max}_{\mu}$, there is a "canonical way" of picking out two factors: The "critical factor" and the "second critical factor".[24] We distinguish cases based on the number of free indices belonging to the second critical factor. Case A corresponds to the case where there are at least two such; in that case, we assume that the indices $_{i_1}, _{i_2}$ belong to the second critical factor. We then introduce a formal operation that erases the index $_{i_1}$, and adds a new derivative free index (denote it by ∇_{i_*}) onto the critical factor. For each $l \in \overline{L}^{Max}_{\mu}$, we will denote the resulting tensor field by $\dot{C}^{l,i_2\ldots i_\mu,i_*}_g$. We also consider the tensor field $\dot{C}^{l,i_2\ldots i_\mu,i_*}_g \nabla_{i_2}\phi_{u+1}$ that is obtained from it by contracting the free index ^{i_2} against a new factor $\nabla_{i_2}\phi_{u+1}$. This new, $(\mu - 1)$-tensor field has a $(u + 1)$-simple character which we will again denote by $\vec{\kappa}'_{simp}$. The claim of Lemma 3.5 is that an equation of the following form holds:

$$\begin{aligned}
&\sum_{l\in\overline{L}^{Max}_{\mu}} a_l X\,div_{i_3} \ldots X\,div_{i_\mu} X\,div_{i_*} \dot{C}^{l,i_2\ldots i_\mu,i_*}_g \nabla_{i_2}\phi_{u+1} \\
&\quad + \sum_{\nu\in N} a_\nu X\,div_{i_2} \ldots X\,div_{i_\mu} C^{\nu,i_1\ldots i_\mu}_g \nabla_{i_1}\phi_{u+1} \\
&\quad + \sum_{p\in P} a_p X\,div_{i_2} \ldots X\,div_{i_{\mu+1}} C^{p,i_1\ldots i_{\mu+1}}_g \nabla_{i_1}\phi_{u+1} \\
&\quad + \sum_{j\in J'} a_j C^{j,i_1}_g \nabla_{i_1}\phi_{u+1} = 0; \qquad (3.3)
\end{aligned}$$

here the tensor fields indexed in N are acceptable and have a simple character $\vec{\kappa}'_{simp}$, but they are doubly subsequent to the tensor fields in the first line. The tensor fields in P have rank $> \mu - 1$ (but they may fail to be acceptable), while the complete contractions in J' are simply subsequent to $\vec{\kappa}'_{simp}$.

[24] In fact, we may have a set of critical factors, and a set of second-critical factors, but for this discussion we will assume that they are unique, for each tensor field indexed in $\overline{L}^{Max}_{\mu}$.

3.3. The rigorous formulation of Lemmas 3.1, 3.2, 3.5

Consider Eq. (3.1). Denote by $L_\mu^{Max} \subset L_\mu$ the index set of the tensor fields of maximal refined double character (recall there may be many maximal refined double characters). A note is in order here: As explained (roughly) in the above discussion, in order to state our lemmas we will be "canonically" picking out some factor, and contracting one of its free indices against a new factor $\nabla\phi_{u+1}$. In particular, for Lemmas 3.1 and 3.2, we will be defining a "critical factor" for the tensor fields in the equation (3.1), while for Lemma 3.5 we will be defining both a "critical factor" and a "second critical factor" for the tensor fields in (3.1). We will make a preliminary note here regarding these notions:

Note on "critical factors": The "critical factor" (or factors) will be universally defined for all the tensor fields or complete contractions in (3.1). In other words, once we specify the critical factor(s), we will be able to examine any tensor field $C_g^{l,i_1\dots i_\beta}$ or complete contraction C_g^j in (3.1) and unambiguously pick out the (set of) critical factor(s) in $C_g^{l,i_1\dots i_\beta}$ or C_g^j. In particular, the critical factor (or set of critical factors) will be defined to be one of the following: Either it will be a factor $\nabla^{(y)}\Omega_a$, for some particular value of a, $1 \leqslant a \leqslant p$, or it will be the curvature factor that is contracting against a given $\nabla\phi_b$, for some chosen value of b, $1 \leqslant b \leqslant u$, or we will define the set of critical factors (in each of the contractions in (3.1)) to stand for the set of curvature factors $\nabla^{(m)}R_{ijkl}$ that are not contracting against *any* factors $\nabla\phi_b$.

In order to avoid confusion further down, we will also remark that the way the critical factor is specified is by examining the μ-tensor fields in (3.1) that have a *maximal* refined double character. Nonetheless, once the critical factor(s) has (have) been specified, we will be able to look at *any* tensor field or complete contraction in (3.1) and pick it (them) out. All this discussion is also true for the "second critical factor" (which will only be defined in the setting of Lemma 3.5).

Rigorous formulation of Lemma 3.1: Our first lemma applies to the case where there are μ-tensor fields in (3.1) for which some factors $S_*\nabla^{(\nu)}R_{ijkl}$ have special free indices (this will be called case I).

In each $C_g^{l,i_1\dots i_\mu}$, $l \in L_\mu^{Max}$,[25] we pick out the factors $T_1 = S_*\nabla^{(\nu)}R_{ijkl}$ with a special free index[26] (observe that by the definition of "maximal" refined double character and by the assumption in the previous paragraph there will be such factors in each tensor field indexed in L_μ^{Max}).

Among those factors, we pick out the ones with the maximum number of free indices (in total). We denote this maximum number of free indices by M. It follows from the definition of *ordering* among refined double characters that this number M is universal among all $C_g^{l,i_1\dots i_\mu}$, $l \in L_\mu^{Max}$.

Definition 3.1. For each $l \in L_\mu^{Max}$ we list all the factors $S_*\nabla^{(\nu)}R_{ijkl}$ which contain a special free index and also have M free indices in total: $\{F_1, \dots, F_\alpha\}$. If at least one of F_h is contracting against at least one factor $\nabla\phi'_b$, then we define the *critical factor* to be the F_h from the list above which is contracting against the factor $\nabla\phi'_b$ with the smallest value for b (say *Min*). If no factors F_h in the above list are contracting against $\nabla\phi'_b$'s, then we define the *critical factor* to be the

[25] $L_\mu^{Max} := \bigcup_{z\in Z_{Max}} L^z$ is the index set of μ-tensor fields of *maximal* refined double character in (3.1).

[26] Recall that a special free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is one of the indices ${}_k, {}_l$.

factor from the list above which is contracting against the factor $\nabla\tilde{\phi}_r$ for the smallest value for r (say Min).

We then denote by $L^{Max}_{\mu,Min} \subset L^{Max}_{\mu}$ the index set of the tensor fields of maximal refined double character $C^{l,i_1\dots i_\mu}_g$ for which the factor $S_*\nabla^{(\nu)}R_{ijkl}$ that is contracting against $\nabla\phi'_{Min}$ or $\nabla\tilde{\phi}_{Min}$, respectively, contains M free indices in total and one of them is special.

Let us observe that there exists some subset $Z'_{Max} \subset Z_{Max}$ so that:

$$L^{Max}_{\mu,Min} = \bigcup_{z\in Z'_{Max}} L^z. \tag{3.4}$$

Now, with no loss of generality (only for notational convenience), we assume that for each $l \in L^{Max}_{\mu,Min}$, the critical factor $S_*\nabla^{(\nu)}R_{ijkl}$ against which $\nabla\tilde{\phi}_{Min}$ contracts has the index ${}_k$ being the free index ${}_{i_1}$. We also recall that $\vec{\kappa}_{simp}$ is the simple character of the tensor fields indexed in L_μ. We will then denote by $\vec{\kappa}^z$ the refined double character of each $C^{l,i_1\dots i_\mu}_g$, $l \in L^z$, $z \in Z'_{Max}$.

Under the assumptions above, our claim is the following:

Lemma 3.1. *Assume* (3.1)*, with weight $-n$, real length σ, $u = \Phi$ and $\sigma_1 + \sigma_2$ factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$; assume also that the tensor fields of maximal refined double character are not "forbidden" (see Definition* 2.12*). Suppose that there are μ-tensor fields in* (2.3) *with at least one special free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$. We then claim that there is a linear combination of acceptable tensor fields,*

$$\sum_{p\in P} a_p C^{p,i_1\dots i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

each with $b \geqslant \mu+1$, with a simple character $\vec{\kappa}_{simp}$ and where each $C^{p,i_1\dots i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ has the property that the free index ${}_{i_1}$ is the index ${}_k$ in the critical factor $S_*\nabla^{(\nu)}R_{ijkl}$ against which $\nabla\tilde{\phi}_{Min}$ is contracting, so that modulo complete contractions of length $\geqslant \sigma+u+2$:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l X\,div_{i_2}\dots X\,div_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&\quad+\sum_{\nu\in N} a_\nu X\,div_{i_2}\dots X\,div_{i_\mu} C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&\quad-\sum_{p\in P} a_p X\,div_{i_2}\dots X\,div_{i_b} C^{p,i_1\dots i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&=\sum_{t\in T} a_t C^{t,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.
\end{aligned} \tag{3.5}$$

Here each $C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ is acceptable and has a simple character $\vec{\kappa}_{simp}$ (and ${}_{i_1}$ is again the index ${}_k$ in the critical factor $S_*\nabla^{(\nu)}R_{ijkl}$), but also has either strictly fewer than M free indices in the critical factor or is doubly subsequent to each $\vec{\kappa}^z$, $z \in Z'_{Max}$. Also, each $C^{t,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$ is in the form (2.1) *and is either simply subsequent* *to $\vec{\kappa}_{simp}$ or $C_g^{t,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ has a u-simple character $\vec{\kappa}_{simp}$ but the index $_{i_*}$ is not a special index. All complete contractions have the same weak $(u+1)$-simple character.*

If we can prove the above then (as we will show in the next subsection) by iterative repetition we can reduce ourselves to proving Proposition 2.1 with the extra assumption that for each $l \in L_\mu$ there are no special free indices in any factor $S_*\nabla^{(\nu)}R_{ijkl}$. Lemma 3.2 will apply to this subcase.

Rigorous formulation of Lemma 3.2: We now assume that all μ-tensor fields in (3.1) have no special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$, but certain μ-tensor fields do have special free indices in factors $\nabla^{(m)}R_{ijkl}$—this will be called case II. We will then pick out those μ-tensor fields in (3.1) that have special free indices in factors $\nabla^{(m)}R_{ijkl}$. (If there are no such tensor fields, we may proceed to Lemma 3.5.)

In order to state our lemma we will need to define the critical factor (or set of critical factors) for the tensor fields appearing in (3.1), in this setting:

Definition 3.2. Firstly we consider the case where there are factors $\nabla^{(m)}R_{ijkl}$ with two special free indices among the μ-tensor fields in (3.1). We will define the critical factor(s) in that setting:

Among all the μ-tensor fields with maximal refined double characters in (3.1), we pick out all the factors $\nabla^{(m)}R_{ijkl}$ with two special free indices. Among those factors, we pick out the ones with the maximal total number of free indices, say $M \geqslant 2$. Denote that list by $\{T_1,\dots,T_\pi\}$. (All the T_i's are in the form $\nabla^{(m)}R_{ijkl}$.) We inquire whether any of the factors $T_1,\dots,T_\pi$ are contracting against a factor $\nabla\phi_h$. If so, we define the critical factor to be the T_i that is contracting against the $\nabla\phi_o$ for the smallest o. If none of the factors $T_1,\dots,T_\pi$ are contracting against a factor $\nabla\phi_h$ we define the set of critical factors to be the set of factors $\nabla^{(m)}R_{ijkl}$ which are not contracting against any factor $\nabla\phi_h$.

The same definition can be applied to define a critical factor in the case where there are no factors $\nabla^{(m)}R_{ijkl}$ with two special free indices but there are factors $\nabla^{(m)}R_{ijkl}$ with one free index (list them out as $\{T_1,\dots,T_{\pi'}\}$ and proceed as above).

Now, we index the maximal μ-tensor fields with M free indices in the critical factor in the index set $\bigcup_{z\in Z'_{Max}} L^z \subset L_\mu$.

Definition 3.3. For each $z \in Z'_{Max}$ define $I_{*,l} \subset I_l$ (recall that I_l stands for the set of free indices in the tensor field $C_g^{l,i_1\dots i_\mu}$) to be the set of special free indices that belong to the critical factor (if it is unique), or to one of the critical factors.

Note: By virtue of the definition of the maximal refined double characters and of the critical factors, we observe that for any two $l_1,l_2 \in \bigcup_{z\in Z'_{Max}} L^z$, we will have $|I_{*,l_1}| = |I_{*,l_2}|$.

Now, for each $z \in Z'_{Max}$ we consider the $(\mu-1)$-tensor fields in the linear combination

$$\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$$

and we will write them as a linear combination of $(\mu-1)$-tensor fields in the form (2.2) (plus error terms—see below):

For each $l \in L^z$, $z \in Z'_{Max}$ and each $_{i_h} \in I_{*,l}$ (we may assume with no loss of generality that $_{i_h}$ is the index $_i$ in some factor $\nabla^{(m)}R_{ijkl}$), we denote by $\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$

the tensor field that arises from $C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$ by replacing the expression $\nabla^{(m)}_{r_1\dots r_m}R_{i_hjkl}\nabla^{i_h}\phi_{u+1}$ by an expression $S_*\nabla^{(m)}_{r_1\dots r_m}R_{i_hjkl}\nabla^{i_h}\phi_{u+1}$. By the first and second Bianchi identity, it then follows that:

$$\begin{aligned}
&C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\\
&\quad=\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\\
&\qquad+\sum_{t\in T}a_tC_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1},
\end{aligned}\tag{3.6}$$

where each $C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$ has the factor $\nabla\phi_{u+1}$ contracting against a derivative index in a factor $\nabla^{(m)}R_{ijkl}$-see the statement of Lemma 3.2.

We denote by $\vec{\kappa}'_{simp}$ and $\vec{\kappa}^z$ the $(u+1)$-simple character (respectively, the $(u+1,\mu-1)$-refined double character) of the tensor fields $\tilde{C}_g^{l,i_1\dots i_\mu}\nabla_{i_h}\phi_{u+1}$, $l\in L^z$, $z\in Z'_{Max}$. (We observe that for each $l\in L^z$, $z\in Z'_{Max}$ the simple characters of the tensor fields $\tilde{C}_g^{l,i_1\dots i_\mu}\nabla_{i_h}\phi_{u+1}$ will be equal.)

Lemma 3.2. *Assume* (3.1) *with weight $-n$, real length σ, $u=\Phi$ and $\sigma_1+\sigma_2$ factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$. Suppose that no μ-tensor fields have special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$, but some have special free indices in factors $\nabla^{(m)}R_{ijkl}$. In the notation above we claim that there exists a linear combination $\sum_{d\in D}a_dC_g^{d,i_1\dots i_b}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ of acceptable tensor fields with a $(u+1)$-simple character $\vec{\kappa}'_{simp}$ and rank $\geqslant\mu$, so that:*

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}X\,div_{i_1}\dots\widehat{X\,div}_{i_h}\dots X\,div_{i_\mu}\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\\
&\qquad+\sum_{\nu\in N}a_\nu X\,div_{i_2}\dots X\,div_{i_\mu}C_g^{\nu,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&\qquad+\sum_{d\in D}a_dX\,div_{i_1}\dots X\,div_{i_b}C_g^{d,i_1,\dots,i_b}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\\
&\quad=\sum_{t\in T}a_tC_g^{t,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_u)\nabla_{i_*}\phi_{u+1},
\end{aligned}\tag{3.7}$$

where the $(\mu-1)$-tensor fields $C_g^{\nu,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ are acceptable, have $(u+1)$-simple character $\vec{\kappa}'_{simp}$ but also either have fewer than M free indices in the factor against which $\nabla_{i_h}\phi_{u+1}$ contracts,[27] or are doubly subsequent to all the refined double characters $\vec{\kappa}^z$, $z\in Z'_{Max}$. Moreover we require that each $C_g^{\nu,i_1\dots,i_\mu}$ has the property that at least one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ is neither free nor contracting against a factor $\nabla\phi'_h$, $h\leqslant u$. The complete contractions $C_g^{t,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_u)\nabla_{i_*}\phi_{u+1}$ are in the form (2.1) *and are simply subsequent to $\vec{\kappa}'_{simp}$.*

[27] "Fewer than M free indices" where we also count the free index ${}_{i_h}$.

We will show in Section 4 that if we can prove the above, then we can reduce ourselves to proving Proposition 2.1 when none of the μ-tensor fields in the lemma hypothesis have special free indices. Lemma 3.5 will apply to that setting:

Rigorous formulation of Lemma 3.5: Recall that we have grouped up the μ-tensor fields $C^{l,i_1\dots i_\mu}_g$ according to their refined double characters: $\sum_{l\in L_\mu} a_l C^{l,i_1\dots i_\mu}_g = \sum_{z\in Z}\sum_{l\in L^z} a_l \dots$. We have then picked out the sublinear combinations in $\sum_{l\in L_\mu} a_l C^{l,i_1\dots i_\mu}_g$ which consist of tensor fields with the same *maximal* refined double character: Thus we obtained a sublinear combination $\sum_{z\in Z_{Max}}\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g$.

Now, in order to state our lemma we will distinguish two further subcases; first we must introduce some more terminology.

We will again define the critical factor for the tensor fields in (3.1):

Definition 3.4. Consider all the μ-tensor fields of maximal refined double character in (3.1), and let M stand for the maximum number of free indices that can belong to a given factor in such a μ-tensor field (we call these "maximal" μ-tensor fields). We then list all the factors that appear with M free indices in some maximal μ-tensor field in (3.1): $\{T_1,\dots,T_\pi\}$. If at least one of those factors T_l is of the form $\nabla^{(p)}\Omega_h$, we define the critical factor to be the $\nabla^{(p)}\Omega_h$ in the list $\{T_1,\dots,T_\pi\}$ with the smallest value h. If none of the factors in that list are in the form $\nabla^{(p)}\Omega_h$, we inquire whether any factors in the list are contracting against factors $\nabla\phi_h$ (or $\nabla\tilde{\phi}_h$). If so, we define the critical factor in (3.1) to be in the list $\{T_1,\dots,T_\pi\}$ that is contracting against the $\nabla\phi_h$ (or $\nabla\tilde{\phi}_h$) with the smallest value of h. Finally, if none of the factors in the list $\{T_1,\dots,T_\pi\}$ are contracting against a $\nabla\phi_h$ (or $\nabla\tilde{\phi}_h$) (so all of them must be in the form $\nabla^{(m)}R_{ijkl}$), then we declare the set of critical factors to be the set of factors $\nabla^{(m)}R_{ijkl}$ that are not contracting against any $\nabla\phi_h$ to be critical factors.

In addition to the critical factor, we now define the *second critical factor* in (3.1). The definition goes as follows:

Definition 3.5. Consider any of the maximal μ-tensor fields in (3.1), $C^{l,i_1\dots i_\mu}_g$, $l\in\bigcup_{z\in Z_{Max}}L^z$. If the critical factor is unique, we construct a list of all the non-critical factors T that belong to one of the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in\bigcup_{z\in Z'_{Max}}L^z$. Suppose that list is $\{T_1,\dots,T_\pi\}$.

We then pick out the second critical factor from that list in the same way that we pick out the critical factor in Definition 3.4.

If there are multiple critical factors, we just define the set of second critical factors to be the set of critical factors.

In either case, we denote by M' the total number of free indices that belong to the (a) second critical factor.

Now, an important note: The "critical factor" (or factors) in (3.1) has been defined based on the maximal refined double characters $\vec{L}^z$, $z\in Z_{Max}$. Nonetheless, once we have chosen a critical factor (or a set of critical factors) for the set $C^{l,i_1\dots i_\mu}_g$, $l\in\bigcup_{z\in Z'_{Max}}L^z$, we may then unambiguously speak of the critical factor(s) for *all* the tensor fields and complete contractions appearing in (3.1).

The two cases for Lemma 3.5: We now distinguish two cases on (2.3): We say that (2.3) (where no tensor fields contain special free indices) fall under case A if $M'\geqslant 2$. It falls under case B if $M'\leqslant 1$.

Now, we will state Lemma 3.5 after we first state a few extra claims. These claims will be proven in the paper [7] in this series.[28]

The extra claims needed to state Lemma 3.5: In order to state Lemma 3.5, we must first show some preliminary results. We introduce some definitions:

We denote by $L^*_\mu \subset L_\mu$ the index set of those tensor fields $C^{l,i_1\dots i_\mu}_g$ in (3.1) for which some chosen factor $\nabla^{(A)}_{r_1\dots r_A}\Omega_x$ (the value x which determines this factor will be chosen at a later stage; we may also *not* choose any such factor $\nabla^{(A)}_{r_1\dots r_A}\Omega_x$, in which case we set $L^*_\mu = \emptyset$) has $A=2$ and both indices $_{r_1}, _{r_2}$ are free indices.

Also, we define $L^+_\mu \subset L_\mu$ to stand for the index set of those μ-tensor fields that have a free index ($_{i_\mu}$ say) belonging to a factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_h$ (without derivatives) and in fact $_j = _{i_\mu}$.

We also denote by

$$\sum_{l\in\tilde{L}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

a linear combination of acceptable μ-tensor fields with simple character $\vec{\kappa}_{simp}$ which do not have special free indices and does not contain tensor fields in any of the above two forms.

Lemma 3.3. *Assume* (3.1)*, where the terms in the LHS of that equation have weigh $-n$, real length σ, Φ factors $\nabla\phi, \nabla\phi', \nabla\tilde{\phi}$ and $\sigma_1+\sigma_2$ curvature factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$*[29]*; assume also that no μ-tensor field there has any special free indices. We claim that there is a linear combination of acceptable $(\mu+1)$-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ with a simple character $\vec{\kappa}_{simp}$ so that:*

$$\begin{aligned}
&\sum_{l\in L^*_\mu\cup L^+_\mu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad+\sum_{p\in P} a_p X\,div_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&=\sum_{j\in J} a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad+\sum_{l\in\tilde{L}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}\tag{3.8}$$

modulo complete contractions of length $\geqslant \sigma+u+\mu+1$. The tensor fields indexed in J on the right-hand side are simply subsequent to $\vec{\kappa}_{simp}$.

Assuming the above lemma, by making the $\nabla\upsilon$'s into $X\,div$s (see the last lemma in the Appendix of [2]) and replacing into (3.1) we are reduced to showing our Proposition 2.1 under the

[28] These claims involve much notation and are rather technical. The reader may choose to disregard them in the first reading, as they are not central to the argument.

[29] See the discussion on the *induction* in Section 3.1.

additional assumption that $L^*_\mu \cup L^+_\mu = \emptyset$. So for the rest of this subsection we will be assuming that $L^*_\mu \cup L^+_\mu = \emptyset$.

We now consider the sublinear combination indexed in $L \setminus L_\mu (= L_{>\mu})$ in (3.1). We define $L''_+ \subset L_\mu$ to stand for the index set of tensor fields with a factor $R_{ijkl}\nabla^i\tilde{\phi}_h$ for which *both* indices ${}_j, {}_k$ are free.

Now, we denote by

$$\sum_{l\in\tilde{L}'} a_l C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

a generic linear combination of acceptable $(\mu+1)$-tensor fields that do not contain tensor fields in the form described above. We claim:

Lemma 3.4. *Assume* (3.1) *with weight $-n$, real length σ, $u = \Phi$ and $\sigma_1+\sigma_2$ factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$*[30]*; assume also that none of the μ-tensor fields have special free indices, and that $L^*_\mu \cup L^+_\mu = \emptyset$. We claim that there exists a linear combination of acceptable* $(\mu+2)$*-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ with simple character $\vec{\kappa}_{simp}$, so that:*

$$\begin{aligned}
&\sum_{l\in L''_+} a_l X\,div_{i_1}\dots X\,div_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\quad+\sum_{p\in P} a_p X\,div_{i_1}\dots X\,div_{i_{\mu+2}} C^{p,i_1\dots i_{\mu+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&=\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\quad+\sum_{l\in\tilde{L}'} a_l X\,div_{i_1}\dots X\,div_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),
\end{aligned} \tag{3.9}$$

modulo complete contractions of length $\geqslant \sigma+u+1$. Here $\sum_{j\in J}\dots$ stands for a linear combination of complete contractions that are simply subsequent to $\vec{\kappa}_{simp}$.

We observe that if we can show the above, then replacing into (3.1) we are reduced to proving Proposition 2.1 under the extra assumptions that $L^*_\mu \cup L^+_\mu \cup L''_+ = \emptyset$. So for the rest of this section we will be making that assumption. The proof of these two lemmas is given in the paper [7] in this series.

Notation and language conventions for Lemma 3.5: Recall the two cases A, B. We will first formulate our claim in case A (where $M' \geqslant 2$). We introduce some notation.

We define $Z'_{Max} \subset Z_{Max}$ as follows[31]: $z \in Z'_{Max}$ if and only if $C^{l,i_1\dots i_\mu}_g$, $l \in L^z$ has M' free indices in the second critical factor (see Definition 3.5).

Now, we first consider the case where there is a unique second critical factor in (3.1). For each $l \in L^z$, $z \in Z'_{Max}$, we assume with no loss of generality that the indices ${}_{i_1}, {}_{i_2}$

[30] See the discussion in Section 3.1.

[31] Recall that $\bigcup_{z\in Z_{Max}} L^z \subset L_\mu$ stands for the index set of the μ-tensor fields.

belong to the second critical factor, and that the index i_1 is a derivative index (the second assumption can be made since all μ-tensor fields in (3.1) have no special free indices now; hence if two free indices belong to the same factor, one of them must be a derivative index). We then denote by $\dot{C}_g^{l,i_2\ldots i_\mu,i_*}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)$ the tensor field that formally arises from $C_g^{l,i_1\ldots i_\mu}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)$ by erasing the free index i_1 from the second critical factor and adding a derivative index ∇_{i_*} onto the critical factor, and making the index i_* free. We denote by $\vec{L}^{z,\sharp}$ the $(u+1,\mu-1)$-refined double character of these $\dot{C}_g^{l,i_2\ldots i_\mu,i_*}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_2}\phi_{u+1}, l\in L^z, z\in Z'_{Max}$.

Now, the case where there are multiple second critical factors: If there are $k>1$ second critical factors $T_1,\ldots,T_k$ in (3.1), then for each $C_g^{l,i_1\ldots i_\mu}$, $l\in L^z, z\in Z'_{Max}$ we denote by $\{i_1,\ldots,i_\alpha\},\{i_{\alpha+1},\ldots,i_{2\alpha}\},\ldots,\{i_{(k-1)\alpha+1},\ldots,i_{k\alpha}\}$ the set of free indices that belong to $T_1,\ldots,T_k$ respectively. We will be making the assumption (with no loss of generality, for the reason explained in the previous paragraph) that the index $i_{t\alpha+1}$ is a derivative index for every $t=0,1,\ldots,k-1$. We then denote by $\dot{C}_g^{l,i_1\ldots\hat{i}_{t\alpha+1}\ldots i_\mu,i_*}$ the tensor field that arises from $C_g^{l,i_1\ldots i_\mu}$ by erasing the index $i_{t\alpha+1}$ and adding a free derivative index i_* onto the (a) critical factor (and adding, if there are multiple critical factors).

In both cases above we define $\vec{\kappa}^+_{simp}=Simp(\vec{L}^{z,\sharp})$, for some $z\in Z'_{Max}$ (notice that the definition is independent of the element $z\in Z'_{Max}$).

A note is needed regarding this definition: In the case where the set of critical and second critical factors coincide, then when we "add a free derivative index i_* onto (a) critical factor", we will be adding it on any critical factor other than the one to which $i_{t\alpha+2}$ belongs. Observe that for any $l=0,1,\ldots,k$ the tensor fields $\dot{C}_g^{l,i_1\ldots\hat{i}_{t\alpha+1}\ldots i_\mu,i_*}\nabla_{t\alpha+2}\phi_{u+1}$ have the same $(u+1,\mu-1)$-refined double character, which we again denote by $\vec{L^{z,\sharp}}$, $z\in Z'_{Max}$ (as in the case of a unique second critical factor).

One last language convention: For uniformity, in case A of Lemma 3.5 we will call the (set of) second critical factor(s) the *(set of) crucial factor(s);* in case B of Lemma 3.5 we will call the (set of) critical factor(s) the *(set of) crucial factor(s)*.

Our claim is then the following:

Lemma 3.5. *Assume* (3.1) *with weight $-n$, real length σ, $u=\Phi$ and $\sigma_1+\sigma_2$ factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$, and additionally assume that no μ-tensor field in* (3.1) *has special free indices; assume also that $L^*_\mu\cup L^+_\mu\cup L''_+=\emptyset$ (in the notation of the extra claims above). Recall the cases A, B that we have distinguished above.*

Consider case A: Recall that k stands for the (universal) number of second critical factors among the tensor fields indexed in $\bigcup_{z\in Z'_{Max}}L^z$. Recall also that for each $z\in Z'_{Max}$ α is the number of free indices in the (each) second critical factor. We claim that:

$$\binom{\alpha}{2}\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{r=0}^{k-1}X\,div_{i_2}\ldots X\,div_{i_*}\dot{C}_g^{l,i_1\ldots\hat{i}_{r\alpha+1}\ldots i_\mu,i_*}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_{r\alpha+2}}\phi_{u+1}$$
$$+\sum_{\nu\in N}a_\nu X\,div_{i_2}\ldots X\,div_{i_\mu}C_g^{\nu,i_1\ldots i_\mu}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+\sum_{t\in T_1}a_t X\,div_{i_1}\ldots X\,div_{i_{z_t}}C_g^{t,i_1\ldots i_{z_t}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_{u+1})$$

$$
\begin{aligned}
&+\sum_{t\in T_2} a_t X\,div_{i_2}\dots X\,div_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&+\sum_{t\in T_3} a_t X\,div_{i_1}\dots X\,div_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\
&+\left(\sum_{t\in T_4} a_t X\,div_{i_1}\dots X\,div_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\right)\\
=&\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0, \qquad (3.10)
\end{aligned}
$$

modulo complete contractions of length $\geqslant \sigma+u+2$*. Here each* $C_g^{\nu,i_1\dots i_\mu}$ *is acceptable and has a simple character* $\vec{\kappa}^+_{simp}$ *and a double character that is doubly subsequent to each* $\vec{L}^{z,\sharp}$*,* $z\in Z'_{Max}$*.*

$$\sum_{t\in T_1} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

is a generic linear combination of acceptable tensor fields with a $(u+1)$*-simple character* $\vec{\kappa}^+_{simp}$*, and with* $z_t\geqslant\mu$*.*

$$\sum_{t\in T_2} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

$(z_t\geqslant\mu+1)$ *is a generic linear combination of acceptable tensor fields with a u-simple character* $\vec{\kappa}_{simp}$*, with the additional restriction that the free index* ${}_{i_1}$ *that belongs to the (a) crucial factor*[32] *is a special free index.*[33]

Now, $\sum_{t\in T_2} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *is a generic linear combination of acceptable tensor fields with* $(u+1)$*-simple character* $\vec{\kappa}^+_{simp}$ *and* $z_t\geqslant\mu$*,*[34] *and moreover one unacceptable factor* $\nabla\Omega_h$ *which does not contract against any factor* $\nabla\phi_t$*.*

The sublinear combination $\sum_{t\in T_4}\dots$ *appears only if the second critical factor is of the form* $\nabla^{(B)}\Omega_k$*, for some k. In that case,* $t\in T_4$ *means that there is one unacceptable factor* $\nabla\Omega_k$*, and it is contracting against a factor* $\nabla\phi_r$*:* $\nabla_i\Omega_k\nabla^i\phi_r$*, and moreover if* $z_t=\mu$ *then one of the free indices* ${}_{i_1},\dots,{}_{i_\mu}$ *is a derivative index, and if it belongs to* $\nabla^{(B)}\Omega_h$ *then* $B\geqslant 3$*.*

Finally,

$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

stands for a generic linear combination of complete contractions that are u-simply subsequent to $\vec{\kappa}_{simp}$*.*

In case B, we just claim the whole of Proposition 2.1.

---

[32] I.e. the second critical factor, in this case.

[33] Recall that a special free index is either an index ${}_{k,l}$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ or an internal index in a factor $\nabla^{(m)}R_{ijkl}$.

[34] If $z_t=\mu$ then we additionally claim that $\nabla\phi_{u+1}$ is contracting against a derivative index, and if it is contracting against a factor $\nabla^{(B)}\Omega_h$ then $B\geqslant 3$; moreover, in this case $C_g^{t,i_1\dots i_\mu}$ will contain no special free indices.

Note: Lemmas 3.1, 3.2, 3.5 (and also Lemmas 3.3, 3.4) will be proven in the final paper [7] in this series. In the remainder of the present paper we will show that these three lemmas imply the inductive step of Proposition 2.1.

4. Proof that Proposition 2.1 follows from Lemmas 3.1, 3.2, 3.5 (and Lemmas 3.3, 3.4)

4.1. Introduction

General discussion: In this section we will show how the inductive step of Proposition 2.1 (see the discussion in the beginning of the last section) follows from Lemmas 3.1–3.5 (apart from certain *special cases* where we will prove the inductive step of Proposition 2.1 *directly*, without using Lemmas 3.1, 3.2 and 3.5). We stress that in this derivation, we *will be* using the inductive assumption on Proposition 2.1. We also repeat that when we *prove* Lemmas 3.1–3.5, we will again be using the inductive assumptions on Proposition 2.1.

We will prove this assertion by distinguishing three cases regarding the *assumption* of Proposition 2.1 (recall that the assumption is Eq. (3.1)). The cases we distinguish are based on the maximal refined double characters among the μ-tensor fields in (3.1):

Recall that the (u,μ)-refined double characters $\vec{L}^z, z \in Z'_{Max}$ are among the maximal (u,μ)-refined double characters in (3.1). We recall cases I, II, III as follows: If for any $\vec{L}^z, z \in Z'_{Max}$ there is a special free index in some factor $S_*\nabla^{(\nu)}R_{ijkl}$, then we declare that (3.1) falls under case I of Proposition 2.1.[35] If for $\vec{L}^z, z \in Z'_{Max}$ there are no special free indices in any factor of the form $S_*\nabla^{(\nu)}R_{ijkl}$ but there are special free indices in factors of the form $\nabla^{(m)}R_{ijkl}$, then we declare that (3.1) falls under case II of Proposition 2.1.[36] Finally, if there are no special free indices at all in any $\vec{L}^z, z \in Z'_{Max}$, then we declare that (3.1) falls under case III of Proposition 2.1.[37]

In the remainder of this paper we will show that in case I, Lemma 3.1 implies Proposition 2.1. In case II, Lemma 3.2 implies Proposition 2.1, while in case III Lemma 3.5 (and Lemmas 3.3, 3.4) imply Proposition 2.1.

More precisely, we will show that in the setting of each of Lemmas 3.1, 3.2, 3.5 it follows that for each $z \in Z'_{Max}$ there is a linear combination of acceptable $(\mu+1)$-tensor fields (indexed in P below) with a (u,μ)-double character $\vec{L}^z$ so that:

$$\begin{aligned}
&\sum_{l\in L^z} a_l C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon \\
&\qquad - X\,div_{i_{\mu+1}}\sum_{p\in P} a_p C_g^{p,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon \\
&\quad = \sum_{t\in T} a_t C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned} \tag{4.1}$$

where each $C_g^{t,i_1\dots i_\mu}$ is subsequent (simply or doubly) to $\vec{L}^z$.

[35] Observe that if this property holds for one of the maximal refined double characters $\vec{L}^z, z \in Z'_{Max}$, it will then hold for all of them.

[36] The observation of the above footnote still holds.

[37] The observation of the above footnote still holds.

Let us just observe how (4.1) implies Proposition 2.1: Firstly, (4.1) shows us that the conclusion of Proposition 2.1 holds for the sublinear combination indexed in $\bigcup_{z\in Z'_{Max}} L^z \subset \bigcup_{z\in Z_{Max}} L^z$. But then we only have to make the $\nabla\upsilon$'s into $X\,div$'s in the above[38] and substitute back into (2.3) and we will be reduced to proving our Proposition 2.1 assuming an equation:

$$\begin{aligned}
&\sum_{z\in Z_{Max}\setminus Z'_{Max}} \sum_{l\in L^z} a_l X\,div_{i_1}\dots X\,div_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\quad+\sum_{t\in T} a_t X\,div_{i_1}\dots X\,div_{i_\mu} C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\quad+\sum_{l\in L_{\beta>\mu}} a_l X\,div_{i_1}\dots X\,div_{i_\beta} C^{l,i_1\dots i_\beta}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&=\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),
\end{aligned} \tag{4.2}$$

where the tensor fields indexed in T are acceptable μ-tensor fields which are (doubly) subsequent to the tensor fields in the first line. But then our Proposition 2.1 follows by induction, since there are finitely many (u,μ)-refined double characters.

Technical discussion of the difficulties in deriving (4.1) from Lemmas 3.1, 3.2, 3.5: Let us observe that *at a rough level*, it would seem that the conclusions of Lemmas 3.1, 3.2 and 3.5 would fit into the inductive assumption of Proposition 2.1 because the weight is $-n$, the real length of the terms in σ but we have *increased* the number Φ of factors $\nabla\phi$, $\nabla\tilde{\phi}$, $\nabla\phi'$. Hence, if that were true, one could hope that a direct application of Corollary 1 to the conclusions of these lemmas would imply Eq. (4.1). Unfortunately, this is not quite the case, for the reasons we will explain in the next three paragraphs. Therefore, there is some manipulation to be done with the conclusions of Lemmas 3.1, 3.2 and 3.5 in order to be able to apply the inductive assumption of Proposition 2.1 (and hence also of Corollary 1), and this manipulation will be done in the remainder of this paper. The obstacles to directly applying the inductive assumption of Proposition 2.1 to the conclusions of Lemmas 3.1, 3.2 and 3.5 are as follows:

Lemma 3.1: Here the $(\mu-1)$-tensor fields $C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$ in Eq. (3.5) have the factor $\nabla\phi_{u+1}$ contracting against the index $_k$ of a factor $S_*\nabla^{(\nu)}R_{ijkl}$. Thus, they are *not* of the form (2.2). Therefore, the inductive assumption of Proposition 2.1 cannot be directly applied to (3.5).

Lemma 3.2: Here the $(\mu-1)$-tensor fields in (3.7) *are* acceptable in the form (2.2), and the inductive assumption of Proposition 2.1 *can* be applied to (3.7). Nonetheless, if we directly apply the inductive assumption of Proposition 2.1 to (3.7), we will obtain an equation similar to (2.4), but involving a linear combination

$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} \tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_h}\upsilon\dots\nabla_{i_\mu}\upsilon$$

[38] See the last lemma in the Appendix of [2].

rather than a linear combination

$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$

as required (the important difference here is the symbol $\tilde{}$, which stands for an S_*-symmetrization). In other words, in $\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$ some factor $\nabla^{(m)}R_{ijkl}$ has been S_*-symmetrized. It is then not obvious how to manipulate this equation to obtain (2.4).

Lemma 3.5: In this case there are numerous obstacles to deriving the inductive step of Proposition 2.1 from (3.10). Firstly, the tensor fields indexed in T_3, T_4 are *not* acceptable. Secondly, even if these index sets were empty, the tensor fields indexed in T_2 *do not* have the $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ of the tensor fields in the first line of (3.10). Lastly, even if this index set T_2 were also empty, and we directly applied the inductive assumption of Proposition 2.1 to (3.10), we would obtain a statement involving the expression:

$$\binom{\alpha}{2}\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{r=0}^{k-1} \dot{C}_g^{l,i_1\dots\hat{i}_{r\alpha+1}\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{r\alpha+2}}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_*}\upsilon, \tag{4.3}$$

and this expression is quite different from the expression we need in (4.1).

4.2. Derivation of Proposition 2.1 in case I from Lemma 3.1

We start this subsection with a technical lemma that will be needed here, but will also be used on multiple occasions throughout this work:

Lemma 4.1. *Let $\sum_{x\in X} a_x C_g^{x,i_1\dots i_\beta}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)$ stand for a linear combination of tensor fields, each with rank β, for some given number β, and with a given simple character $\vec{\kappa}^*_{simp}$, with real length $\sigma \geqslant 4$ and weight $-n$. We also assume that there is a given factor $\nabla^{(y)}\Omega_c$, $y \geqslant 1$ (c independent of x) in each $C_g^{x,i_1\dots i_\beta}$ all of whose indices are contracting against factors $\nabla\phi$.*[39] *We assume an equation:*

$$\sum_{x\in X} a_x X_* div_{i_1}\dots X_* div_{i_\beta} C_g^{x,i_1\dots i_\beta}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b) + \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b) = 0, \tag{4.4}$$

where $X_* div_i$ stands for the sublinear combination in $X div_i$ for which ∇_i is not allowed to hit the factor $\nabla^{(y)}\Omega_c$. The complete contractions in J are simply subsequent to $\vec{\kappa}^*_{simp}$. We additionally

[39] If $y \geqslant 2$ then our tensor fields are assumed to be acceptable. If $y = 1$ then we assume that $\nabla\Omega_c$ is the only unacceptable factor.

assume that if we formally erase the factor $\nabla^{(y)}\Omega_c$ along with the factors $\nabla\phi_h$ that it is contracting against, then the resulting tensor fields are acceptable, and none of them is "forbidden" in the sense of Definition 2.12.

We claim that we can then write:

$$\sum_{x\in X} a_x X\,div_{i_1}\dots X\,div_{i_\beta} C_g^{x,i_1\dots i_\beta}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)$$
$$= \sum_{x\in X'} a_x X\,div_{i_1}\dots X\,div_{i_\beta} C_g^{x,i_1\dots i_\beta}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)$$
$$+ \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b), \tag{4.5}$$

where the tensor fields indexed in X' are exactly like the ones indexed in X, only the chosen factor $\nabla^{(y)}\Omega_c$ has at least one index that is not contracting against a factor $\nabla\phi$. The complete contractions in J are simply subsequent to $\vec{\kappa}^*_{simp}$.

(Sketch of the) Proof of Lemma 4.1. We just apply the eraser to the factor $\nabla^{(y)}\Omega_c$ and the factors $\nabla\phi_h$ that it is contracting against in (4.4), obtaining a new true equation. We can then iteratively apply Corollary 1 to this new true equation, multiplying by $\nabla^{(B)}_{r_1\dots r_B}\Omega_c\nabla^{r_1}\upsilon\dots\nabla^{r_B}\upsilon$ and making all the factors $\nabla\upsilon$ into $X\,div$'s at each stage. This would show our claim except for the caveat that in the last step of the above iteration, we might not be able to apply Corollary 1 if the tensor fields of maximal refined double character are in one of the forbidden forms of Corollary 1 with rank $>\beta$. In that case, in the last step we use Lemma A.2 below (setting $\Phi=\nabla^{(y)}_{r_1\dots r_c}\Omega_c\nabla^{r_1}\phi_{h_1}\dots\nabla^{r_y}\phi_{h_y}$). That concludes the proof of our claim in this case. □

Furthermore, we have a weaker form of Lemma 4.1 when $\sigma=3$. We firstly introduce a definition that will be used on a number of occasions below:

Definition 4.1. Consider any tensor field in the form (2.2). We consider any set of indices, $\{_{x_1},\dots,_{x_s}\}$ belonging to a factor T (here T is not in the form $\nabla\phi$). We assume that these indices are neither free nor are contracting against a factor $\nabla\phi_h$.

If the indices belong to a factor T in the form $\nabla^{(B)}\Omega_1$ then the indices $\{_{x_1},\dots,_{x_s}\}$ are removable provided $B\geqslant s+2$.

Now, we consider indices that belong to a factor $\nabla^{(m)}R_{ijkl}$ (and are neither free nor are contracting against a factor $\nabla\phi_h$). Any such index $_x$ which is a derivative index will be removable. Furthermore, if T has at least two free derivative indices, then if neither of the indices $_{i}, _{j}$ are free then we will say one of $_{i}, _{j}$ is removable; accordingly, if neither of $_{k}, _{l}$ is free then we will say that one of $_{k}, _{l}$ is removable. Moreover, if T has one free derivative index then: if none of the indices $_{i}, _{j}$ are free then we will say that one of the indices $_{i}, _{j}$ is removable; on the other hand if one of the indices $_{i}, _{j}$ is also free and none of the indices $_{k}, _{l}$ are free then we will say that one of the indices $_{k}, _{l}$ is removable.

Now, we consider a set of indices $\{_{x_1},\dots,_{x_s}\}$ that belong to a factor $T=S_*\nabla^{(\nu)}R_{ijkl}$ and are not special, and are not free and are not contracting against any $\nabla\phi$. We will say that these indices are removable if $s\leqslant\nu$. Furthermore, if none of the indices $_{k}, _{l}$ are free and $\nu>0$ and at least one of the other indices in T is free, we will say that one of the indices $_{k}, _{l}$ is removable.

Weaker version of Lemma 4.1 when $\sigma = 3$:

Lemma 4.2. *Assume Eq.* (4.4) *when* $\sigma = 3$ *and assume additionally that every tensor field indexed in* X *has a removable index. Then* (4.5) *still holds.*

Proof. The argument essentially follows the ideas developed in the paper [4]. Firstly, we observe that (possibly after applying the second Bianchi identity) we can explicitly write:

$$\begin{aligned}
&\sum_{x\in X} a_x X\,div_{i_1}\dots X\,div_{i_\beta} C_g^{x,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)\\
&\quad=\sum_{x\in X'} a_x X\,div_{i_1}\dots X\,div_{i_\beta} C_g^{x,i_1\dots i_\beta}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)\\
&\qquad+\sum_{x\in \overline{X}} a_x X\,div_{i_1}\dots X\,div_{i_\gamma} C_g^{x,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)\\
&\qquad+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b),
\end{aligned}\tag{4.6}$$

where the tensor fields indexed in $\overline{X}$ have all the properties of the ones indexed in X, only they all have rank $\gamma \geqslant \beta + 1$, and they also have no removable indices. The sublinear combination $\sum_{x\in X'}\dots$ (here and below, when it appears on the RHS) stands for a *generic* linear combination as described in Lemma 4.1.

We then observe that we can write:

$$\begin{aligned}
&\sum_{x\in \overline{X}} a_x X\,div_{i_1}\dots X\,div_{i_\gamma} C_g^{x,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)\\
&\quad=(Const)_* X\,div_{i_1}\dots X\,div_{i_\gamma} C_g^{*,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)\\
&\qquad+\sum_{x\in X'} a_x X\,div_{i_1}\dots X\,div_{i_\beta} C_g^{x,i_1\dots i_\beta}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)\\
&\qquad+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)
\end{aligned}\tag{4.7}$$

where the tensor field $C_g^{*,i_1\dots i_\gamma}$ is zero unless $\sigma_1 = \sigma_2 = 0$ or $\sigma_1 = 2$ or $\sigma_2 = 2$. In those cases, the tensor field $C_g^{*,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b)$ is, respectively:

$$\begin{aligned}
&pcontr\big(\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t}\Omega_1\otimes\nabla^{(B)}_{j_1\dots j_b y_1\dots y_r}\Omega_2\otimes\nabla^{u_1}\phi_1\otimes\dots\otimes\nabla^{j_b}\phi_f\otimes\nabla^{(B)}_{z_1\dots z_q}\Omega_3\\
&\quad\otimes\nabla^{z_1}\phi_{f+1}\otimes\nabla^{z_q}\phi_{u+1}\big)
\end{aligned}\tag{4.8}$$

(here if $y \geqslant 2$ then $b = 0$; if $y \leqslant 1$ then $y = 2 - b$),

$$\begin{aligned}
&pcontr\big(\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t}R_{i_{a+1}ji_{a+2}l}\otimes\nabla^{(r)}_{y_1\dots y_r}R_{i_{a+3}}{}^{j}{}_{i_{a+4}}{}^{l}\otimes\nabla^{u_1}\phi_1\otimes\dots\otimes\nabla^{j_b}\phi_f\\
&\quad\otimes\nabla^{(B)}_{z_1\dots z_q}\Omega_3\otimes\nabla^{z_1}\phi_{f+1}\otimes\nabla^{z_q}\phi_{u+1}\big),
\end{aligned}\tag{4.9}$$

$$pcontr\big(S_*\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t}R_{i i_{a+1} i_{a+2} l}\otimes\nabla^{(r)}_{y_1\dots y_r}R_{i' i_{a+3} i_{a+4}}{}^{l}\otimes\nabla^{i}\tilde{\phi}_1\otimes\nabla^{i'}\tilde{\phi}_2$$
$$\otimes\nabla^{u_1}\phi_3\otimes\dots\otimes\nabla^{j_b}\phi_f\otimes\nabla^{(B)}_{z_1\dots z_q}\Omega_3\otimes\nabla^{z_1}\phi_{f+1}\otimes\nabla^{z_q}\phi_{u+1}\big). \tag{4.10}$$

Then, picking out the sublinear combination in (4.7) with only factors $\nabla\phi$ contracting against $\nabla^{(B)}\Omega_c$ we derive that $(Const)_*=0$. $\square$

Derivation of Proposition 2.1 (in case I) from Lemma 3.1: Special cases, etc.: Now, we return to the derivation of Proposition 2.1 (in case I) from Lemma 3.1. We will be singling out a further case, which we will call "delicate". In this "delicate" case we will derive Proposition 2.1 from Lemma 3.1 by using an extra lemma (see Lemma 4.3 below). The proof of Lemma 4.3 will be provided in Appendix A to this paper.

The "delicate case": The delicate case is when the μ-tensor fields of maximum refined double character in our lemma assumption have no removable free indices, and moreover their critical factor is in the form: $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{i r_{\nu+1} i_1 l}$, where *all* indices ${}_{r_1},\dots,{}_{r_\nu},{}_{r_{\nu+1}}$ are either free or contracting against a factor $\nabla\phi'_h$.[40]

In that case we have an extra claim, which we will prove in Appendix A:

Lemma 4.3. *For each $z\in Z'_{Max}$, we let $L^z_*\subset L^z$ stand for the index set of tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$ for which the index ${}_l$ in the critical factor contracts against a special index in some factor S_*R_{ijkl}.*

We claim that for each $z\in Z'_{Max}$, we can write:

$$\sum_{l\in L^z_*}a_l C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\sum_{l\in L'^z}a_l C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon. \tag{4.11}$$

Here the terms indexed in L'^z have all the properties of the terms indexed in L^z, but in addition the index ${}_l$ in the critical factor does not contract against a special index in a factor S_*R_{ijkl}.

The derivation of Proposition 2.1 (in case I): Recall the conclusion of Lemma 3.1, Eq. (3.5). Recall that for each tensor field and each complete contraction in (3.5), $\nabla\phi_{u+1}$ is contracting against the crucial factor, which was defined to be the factor $S_*\nabla^{(\nu)}R_{ijkl}$ in $\vec{\kappa}_{simp}$ whose index ${}_i$ is contracting against a chosen factor $\nabla\tilde{\phi}_{Min}$. For notational convenience, we will assume that $Min=1$, i.e. that index ${}_i$ in the crucial factor is contracting against a factor $\nabla\tilde{\phi}_1$.

Define the set *Stan* to stand for the set of numbers o for which the factor $\nabla\phi'_o$ is contracting against one of the indices ${}_{r_1},\dots,{}_{r_{\nu+1}}$ in the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{i r_{\nu+1} kl}$. With no loss of generality, we assume that $Stan=\{2,\dots,q\}$ or $Stan=\emptyset$ (which is equivalent to saying $q=1$- we will be using that convention below).

For convenience, we will assume that for each of the tensor fields appearing in (3.5) the factors $\nabla\phi'_2,\dots,\nabla\phi'_q$ are contracting against the indices ${}_{r_1},\dots,{}_{r_{q-1}}$ in the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{i r_{\nu+1} kl}$.

With this convention, we introduce a new definition:

[40] Notice that by weight considerations and by the definition of maximal refined double character, if one tensor field of maximal refined double character has this property then all of them will.

Definition 4.2. For each tensor field $C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$, $C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$, $C^{p,i_1,\dots,i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ in (3.5), we define $C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$, $C^{\nu,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$, $C^{p,i_2\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$ to stand for the tensor fields that arise by formally replacing the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ir_{\nu+1}kl}\nabla^i\tilde{\phi}_1\nabla^k\phi_{u+1}$ by a factor $\nabla^{(\nu+2)}_{r_1\dots r_\nu r_{\nu+1}l}Y$ (Y is a scalar function).

We observe that the tensor fields we are left with have length $\sigma+u-1$, a factor $\nabla^{(B)}Y$ with $B\geqslant 2$, and are acceptable if we set $Y=\Omega_{p+1}$. We observe that all these tensor fields have the same $(u-1)$-simple character (where we treat the function Y as a function Ω_{p+1}), which we denote by $\tilde{\kappa}_{simp}$.

We also note that each of the tensor fields $C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$, $C^{\nu,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$, $C^{p,i_2\dots i_b}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$ will have the property that the last index $_l$ in $\nabla^{(B)}Y$ (in the tensor field $C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)$) is neither free nor contracting against a factor $\nabla\phi_h$: This is because the last index $_l$ in $\nabla^{(B)}Y$ corresponds to the index $_l$ in the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ir_{\nu+1}kl}\nabla^i\tilde{\phi}_1\nabla^k\phi_{u+1}$ of the tensor field $C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, and this index is neither free nor contracting against any factor $\nabla\phi'_o$ by hypothesis.

We claim an equation:

$$\begin{aligned}&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l X\,div_{i_2}\dots X\,div_{i_\mu}C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\\&\quad+\sum_{\nu\in N}a_\nu X\,div_{i_2}\dots X\,div_{i_\mu}C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\\&\quad-\sum_{p\in P}a_p X\,div_{i_2}\dots X\,div_{i_{\mu+1}}C^{p,i_2\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\\&=\sum_{t\in T'}a_t C^t_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u),\end{aligned}\tag{4.12}$$

which will hold modulo complete contractions of length $\geqslant\sigma+u$. Here the right-hand side stands for a generic linear combination of complete contractions that are simply subsequent to $\tilde{\kappa}_{simp}$.

Proof of (4.12). Since the argument by which we derive this equation will be used frequently in this series of papers, we codify this claim in a lemma:

Lemma 4.4. *Consider a linear combination of acceptable $(\gamma+1)$-tensor fields, $\sum_{x\in X}a_x C^{x,i_1\dots i_{\gamma+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, all in the form* (2.2) *with weight $-n$ and with a given simple character $\bar{\kappa}_{simp}$. Assume that for each of the tensor fields then the index $_{i_1}$ is the index $_k$ in a given factor $S_*\nabla^{(\nu)}R_{ijkl}$, for which the index $_i$ is contracting against a chosen factor $\nabla\tilde{\phi}_w$ (wlog we will assume $w=1$). Assume that $\sum_{z\in Z}a_z C^{z,i_1\dots i_{\epsilon_z+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ is a linear combination with all the features of the tensor fields indexed in X, only now each $\epsilon_z>\gamma$. Assume an equation:*

$$\sum_{x\in X} a_x X\,div_{i_2}\dots X\,div_{i_{\gamma+1}} C_g^{x,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+\sum_{z\in Z} a_z X\,div_{i_2}\dots X\,div_{i_{\epsilon_z}} C_g^{z,i_1\dots i_{\epsilon_z+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$=\sum_{j\in J} a_j C_g^{j,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}; \tag{4.13}$$

here the vector fields in the RHS have a u-weak character Weak($\bar\kappa_{simp}$) and are either simply subsequent to $\bar\kappa_{simp}$ or have one of the two factors $\nabla\phi_1$, $\nabla\phi_{u+1}$ contracting against a derivative index, or both factors $\nabla\phi_w$, $\nabla\phi_{u+1}$ are contracting against antisymmetric indices $_i$, $_j$ or $_k$, $_l$ in some curvature factor.

Denote by $\overline{C}_g^{x,i_2\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_{p+1},\phi_2,\dots,\phi_u)$, $C_g^{z,i_1\dots i_{\epsilon_z+1}}(\Omega_1,\dots,\Omega_{p+1},\phi_2,\dots,\phi_u)$ the tensor fields that arise from $C_g^{x,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_{p+1},\phi_2,\dots,\phi_u)$, $C_g^{z,i_1\dots i_{\epsilon_z+1}}(\Omega_1,\dots,\Omega_{p+1},\phi_2,\dots,\phi_u)$ by formally replacing the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}\nabla^i\tilde\phi_1\nabla^k\phi_{u+1}$ by an expression $\nabla^{(\nu+2)}_{r_1\dots r_\nu jl}\Omega_{p+1}$. Denote by $\tilde\kappa_{simp}$ the $(u-1)$-simple character of each of the resulting tensor fields (they have length $\sigma+u-1$). We then claim that modulo complete contractions of length $\geqslant\sigma+u$:

$$\sum_{x\in X} a_x X\,div_{i_2}\dots X\,div_{i_{\gamma+1}} \overline{C}_g^{x,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_{p+1},\phi_2,\dots,\phi_u)$$
$$+\sum_{z\in Z} a_z X\,div_{i_2}\dots X\,div_{i_{\epsilon_z}} \overline{C}_g^{z,i_1\dots i_{\epsilon_z+1}}(\Omega_1,\dots,\Omega_{p+1},\phi_2,\dots,\phi_u)$$
$$=\sum_{j\in J'} a_j C_g^{j}(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u), \tag{4.14}$$

where the complete contractions indexed in J' are simply subsequent to $\tilde\kappa_{simp}$. We note that the proof of this lemma will be independent of Proposition 2.1.

Note: Before we prove this lemma we remark that by applying it to (3.5) we derive (4.12).

Proof of Lemma 4.4. Denote the left-hand side of (4.13) by F_g. We then denote by F'_g the linear combination that arises from F_g by formally replacing the factors $\nabla_a\phi_1, \nabla_b\phi_{u+1}$ by g_{ab} (the uncontracted metric tensor). Notice that F'_g then consists of complete contractions with one internal contraction in a curvature factor, and with weight $-n+2$ and length $\sigma+u$. Since $F_g=0$ modulo longer complete contractions, and since this equation holds formally, we derive that $F'_g=0$ modulo longer complete contractions. Now, apply the operation *Ricto* Ω_{p+1} to F'_g (see the relevant lemma in the Appendix of [2]). Denote the resulting linear combination by F''_g. By definition of *Ricto* Ω_{p+1}, the minimum length of the complete contractions in F''_g is $\sigma+u-1$. If we denote this sublinear combination by $F''^{\sigma+u-1}_g$ then (virtue of the aforementioned lemma) we will have $F''^{\sigma+u-1}_g=0$ modulo longer complete contractions. By following all the operations we have performed we observe that this equation is precisely (4.14). □

Now, observe that (4.12) falls under our inductive assumption of Proposition 2.1[41]: All the tensor fields are acceptable, and they all have a given simple character $\vec{\kappa}_{simp}$; furthermore, the weight of the complete contractions in (4.12) is $-n+2>-n$. Lastly, recall that we have noted that the last index in $\nabla^{(B)}Y$ is neither free nor contracting against any factor $\nabla\phi_{u+1}$.

We observe that the sublinear combinations of $(\mu-1)$-tensor fields on the left-hand side of (4.12) with maximal double characters are the sublinear combinations:

$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l C_g^{l,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u).$$

(This follows directly from the definition of the maximal refined double characters.) We denote the respective refined double characters for the complete contractions of this form by $\vec{L^z}'$, $z\in Z'_{Max}$. Applying the inductive hypothesis of Corollary 1 to (4.12) and picking out the sublinear combination with a $(u-1,\mu-1)$-double character $Doub(\vec{L^z}')$, we deduce that there is a linear combination of acceptable μ-tensor fields with a refined double character $\vec{L^z}'$,

$$\sum_{r\in R^z} a_r C_g^{r,i_2\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u),$$

such that:

$$\begin{aligned}
&\sum_{l\in L^z} a_l C_g^{l,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad-\sum_{r\in R^z} a_r X\,div_{i_{\mu+1}} C_g^{r,i_2\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&=\sum_{t\in T} a_t C_g^{t,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}\tag{4.15}$$

modulo complete contractions of greater length. (Recall that since we are considering the factor Y to be of the form Ω_{p+1}, each of the tensor fields above has a factor $\nabla^{(b)}Y$, $b\geqslant 2$.)

Here,

$$\sum_{t\in T} a_t C_g^{t,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)$$

stands for a generic linear combination of tensor fields whose refined double character is (simply or doubly) subsequent to the refined double character $\vec{L^z}'$. Moreover, if this tensor field $C_g^{t,i_2\dots i_\mu}$ is only doubly subsequent to $\vec{L^z}'$, then at least one of the indices in the factor $\nabla^{(B)}Y$ is not contracting against a factor of the form $\nabla\upsilon$, $\nabla\phi$. (This follows since those tensor fields arise from the tensor fields indexed in N in (4.12), which have that property by construction.)

[41] Notice that since our assumption (3.1) does not include tensor fields in any of the "forbidden forms", it follows that the tensor fields of minimum rank in (4.12) are also *not* in any of the "forbidden forms".

Now, we refer to (4.15) and we make all the factors $\nabla\upsilon$ that are *not* contracting against the critical factor into an *X div* (see the last lemma in the Appendix of [2]). We thus obtain a new true equation:

$$\sum_{l\in L^z} a_l X\,div_{i_{M+1}}\dots X\,div_{i_\mu} C_g^{l,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_M}\upsilon\dots\nabla_{i_{M-1}}\upsilon$$
$$=\sum_{r\in R^z} a_r X\,div_{i_{M+1}}\dots X\,div_{i_{\mu+1}} C_g^{r,i_2\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon$$
$$+\sum_{t\in T} a_t X\,div_{i_{M+1}}\dots X\,div_{i_\mu} C_g^{t,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon. \quad (4.16)$$

Now, before we can proceed with our argument we will need to employ a technical lemma whose proof we will present in the end of this subsection.

Lemma 4.5. *We claim that we may assume with no loss of generality that for each tensor field indexed in R^z, one index in $\nabla^{(B)}Y$ (the last one, wlog) is not contracting against any factor $\nabla\phi_h$ or $\nabla\upsilon$.*

Under this extra assumption, we can now derive the claim of Proposition 2.1:

Now, since all the tensor fields in L^z, R^z have the same $(u-1,\mu-1)$-double character, $\vec{L^z}'$, it follows that for each tensor field appearing in (4.15) there is a fixed number τ of factors $\nabla\upsilon$ contracting against the factor $\nabla^{(B)}Y$, and also a fixed number $(q-1)$ of factors $\nabla\phi'_o$ contracting against $\nabla^{(B)}Y$. We may assume with no loss of generality that the τ factors $\nabla\upsilon$ are contracting against the first τ indices in $\nabla^{(B)}Y$ and the $q-1$ factors $\nabla\phi_o$ are contracting against the next $q-1$ indices in $\nabla^{(B)}Y$, in a decreasing rearrangement according to the numbers o. By using the Eraser,[42] we can see that under these assumptions, (4.15) will hold formally, subject to the additional feature that for each complete contraction in (4.15), the first $\tau+q-1$ indices are not permuted. (Call this the extra feature.)

In this setting, we define an operation *Replace* that acts on the tensor fields in (4.15) by replacing the factor $\nabla^{(b)}_{r_1\dots r_b}Y$ (recall $b\geqslant 2$) by an expression $S_*\nabla^{(b-2)}_{r_1\dots r_{b-2}}R_{ir_{b-1}kr_b}\nabla^i\phi_1\nabla^k\upsilon$. We denote the resulting tensor fields by:

$$\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon,$$
$$\tilde{C}_g^{r,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon,$$
$$\tilde{C}_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon.$$

We immediately observe that for each $l\in L^z$:

$$Replace\big[\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\big]$$
$$=C_g^{l,i_1,i_2\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon. \quad (4.17)$$

[42] See the relevant lemma in the Appendix of [2].

We then claim that the vector field

$$\sum_{r\in R} a_r \tilde{C}_g^{r,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$

is the one we need for Proposition 2.1. In order to see this, we only have to recall that (4.15) holds formally. We then "memorize" the sequence of formal applications of the identities in Definition 6 in [1], by which we can make the linearization of (4.15) formally equal to the linearization of the right-hand side. We recall that an application of the identities in Definition 6 in [1] to the factor $\nabla^{(b)}_{r_1\dots r_b}Y$ (subject to the extra feature) means that we may freely *permute* the indices ${}_{r_{\tau+q}},\dots,{}_{r_b}$.

Now, we will perform the same sequence of applications of the identities in Definition 6 in [1] to the linear combination:

$$\sum_{l\in L^z} a_l \tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon \\ - \sum_{r\in R^z} a_r X\,div_{i_{\mu+1}}\tilde{C}_g^{r,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon. \tag{4.18}$$

We impose one restriction: When we had permuted the indices ${}_{r_1},\dots,{}_{r_b}$ in a factor $\nabla^{(b)}_{r_1\dots r_b}Y$ in (4.15), we now freely permute them again, but also introduce correction terms, by virtue of the Bianchi identities:

$$\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi_1\nabla^k\upsilon - \nabla^{(m)}_{r_1\dots r_m}R_{ilkj}\nabla^i\phi_1\nabla^k\upsilon = \nabla^{(m)}_{r_1\dots r_m}R_{ikjl}\nabla^i\phi_1\nabla^k\upsilon, \tag{4.19}$$

$$\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi_1\nabla^k\upsilon - \nabla^{(m)}_{r_1\dots j}R_{ir_mkl}\nabla^i\phi_1\nabla^k\upsilon = \nabla^{(m)}_{r_1\dots i}R_{r_mjkl}\nabla^i\phi_1\nabla^k\upsilon, \tag{4.20}$$

$$\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi_1\nabla^k\upsilon - \nabla^{(m)}_{r_1\dots l}R_{ijkr_m}\nabla^i\phi_1\nabla^k\upsilon = \nabla^{(m)}_{r_1\dots k}R_{r_mlij}\nabla^i\phi_1\nabla^k\upsilon. \tag{4.21}$$

Hence, we derive that modulo complete contractions of length $\geqslant \sigma+u+\mu+1$:

$$\sum_{l\in L^z} a_l \tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon \\ - \sum_{r\in R^z} a_r X\,div_{i_{\mu+1}}\tilde{C}_g^{r,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon \\ = \sum_{t\in T} a_t \tilde{C}_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\upsilon \\ + \sum_{t\in T'} a_t \tilde{C}_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon, \tag{4.22}$$

where

$$\sum_{t\in T'} a_t \tilde{C}_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon \tag{4.23}$$

stands for the linear combination of correction terms that arises by virtue of the identities (4.19), (4.20), (4.21).

Specifically, the linear combination of correction terms arises by replacing the crucial factor $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi_1\nabla^k\phi_{u+1}$ by one of the expressions on the right-hand sides of (4.19), (4.20), (4.21). We observe that all the correction terms are acceptable tensor fields that are (simply or doubly) *subsequent* to $\vec{L}^{z'}$.

Thus we have proven that under the assumptions of Lemma 3.1, the claim of Proposition 2.1 follows in this case I. $\square$

Proof of Lemma 4.5. We refer to (4.16). We pick out the sublinear combination of terms in that equation where all indices in the function $\nabla^{(B)}Y$ are contracting against a factor $\nabla\upsilon$ or $\nabla\phi'_h$ (denote the index set of tensor fields in R^z with that property by $\overline{R}^z$). We thus obtain a new equation:

$$\sum_{r\in\overline{R}^z} a_r X_*\,div_{i_{M+1}}\dots X_*\,div_{i_{\mu+1}} C_g^{r,i_2\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon$$
$$=\sum_{j\in J} a_j C_g^{j,i_2\dots i_M}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon \tag{4.24}$$

(where the terms indexed in J are simply subsequent to the simple character of the terms in the first line). If $\sigma\geqslant 4$ and *if* our lemma assumption (3.1) *does not* fall under the delicate case,[43] we then apply Lemma 4.1; if $\sigma=3$ and *if* our lemma assumption (3.1) *does not* fall under the delicate case we apply Lemma 4.2.[44] We thus derive:

$$\sum_{r\in\overline{R}^z} a_r X\,div_{i_{M+1}}\dots X\,div_{i_{\mu+1}} C_g^{r,i_2\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon$$
$$=\sum_{r\in\tilde{R}^z} a_r X\,div_{i_{M+1}}\dots X\,div_{i_{\mu+1}} C_g^{r,i_2\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon$$
$$+\sum_{j\in J} a_j C_g^{j,i_2\dots i_M}(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_M}\upsilon, \tag{4.25}$$

where the terms indexed in $\tilde{R}^z$ have all the features of the terms in R^z but in addition have at least one index in $\nabla^{(B)}Y$ *not* contracting against a factor $\nabla\phi_h$ or $\nabla\upsilon$. Replacing the above into (4.16), we may assume that all terms in R^z have at least one index in $\nabla^{(B)}Y$ *not* contracting against a factor $\nabla\phi_h$ or $\nabla\upsilon$.

Proof of (4.25) when (3.1) falls under the delicate case. All the arguments above can be repeated *except for the application of Lemma* 4.1, because in this setting (4.24) might fall under the

[43] We will explain what to do if (4.24) *does* fall under the delicate case below. The fact that (3.1) does not fall under the delicate case ensures that (4.24) does not fall under a forbidden case of Lemma 4.1.

[44] We observe that Lemma 4.2 can be applied, since we are in a non-delicate case hence the tensor fields indexed in R^z which have all indices in $\nabla^{(B)}Y$ contracting against factors $\nabla\upsilon$ must have a removable index.

forbidden cases of Lemma 4.1. On the other hand, we have also imposed a "delicate assumption" on (3.1), which we will utilize now:

Proof that the technical claim follows from (4.15) when (3.1) falls under the "delicate case". Refer to (4.15). We denote by $R^z_{Bad} \subset R^z$ the index set of tensor fields that have the free index ${}_{i_{\mu+1}}$ being a special index in some simple factor $S_*\nabla^{(\nu)}R_{ijkl}$, *and* with all indices in $\nabla^{(B)}Y$ contracting against a factor $\nabla\phi_h$ or $\nabla\upsilon$ (we will call tensor fields with that property "bad"). We observe that (4.25) can again be derived, *provided* that $R^z_{Bad} = \emptyset$ in (4.24) (this is because when $R^z_{Bad} = \emptyset$ there is no danger of falling under a "forbidden case" of Lemma 4.1, by weight considerations). We will now show that we can write:

$$\sum_{r\in R^z_{Bad}} a_r C^{r,i_2\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\nabla_{i_{\mu+1}}\Phi$$
$$= \sum_{r\in R^z_{Not\,Bad}} a_r C^{r,i_2\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\nabla_{i_{\mu+1}}\Phi$$
$$+\sum_{j\in J} a_j C^{j,i_2\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\nabla_{i_{\mu+1}}\Phi, \tag{4.26}$$

where the terms in $R^z_{Not\,Bad}$ are in the form described in (4.15) and in addition are not bad. The terms indexed in J are simply subsequent to $\vec{\kappa}_{simp}$. If we can show the above, then by making the $\nabla\Phi$ into an *X div* and replacing into (4.15), we are reduced to the case $R^z_{Max} = \emptyset$. We have then noted that the proof above goes through. Thus matters are reduced to showing (4.26).

Proof of (4.26). We may assume with no loss of generality that the free index ${}_{i_{\mu+1}}$ is the index ${}_k$ in a factor $S^\sharp\nabla^{(\rho)}_{r_1\dots r_\rho}R_{ijkl}$, where $S^\sharp$ stands for the symmetrization over the index ${}_l$ and all the indices in the above that are not contracting against a factor $\nabla\upsilon$ or $\nabla\phi'_h$ (the correction terms that we obtain from this $S^\sharp$-symmetrization would be tensor fields which are not "bad"—as allowed in (4.26)). We then pick out the sublinear combination of terms in (4.15) with a factor $\nabla^{(B)}Y$ that has all its indices *except* one (say the index ${}_s$) contracting against a factor $\nabla\phi_h$ or $\nabla\upsilon$, and the index ${}_s$ contracting against a special index in a factor $S_*\nabla^{(\rho)}R_{ijkl}$. By virtue of the "delicate assumption", this sublinear combination will be of the form:

$$\sum_{r\in R^z_{BAD}} a_r\, Hit_Y\, div_{i_{\mu+1}} C^{r,i_2\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$+\sum_{j\in J} a_j C^{j,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_2,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon. \tag{4.27}$$

Then, we consider the first conformal variation $Image^1_X[F_g]$ of (4.15) and we pick out the terms where one of the factors $\nabla\phi_h$, $h\in Def(\vec{\kappa}_{simp})$ is contracting against the factor $\nabla^{(B)}Y$, *for which we additionally require that all other indices contract against $\nabla\upsilon$'s or $\nabla\phi$'s*. The above sublinear combination must vanish separately. We thus derive a new equation:

$$\sum_{r\in R^z_{BAD}} a_r\, Hit_Y\, div_{i_{\mu+1}} Op[C]_g^{r,i_2\ldots i_{\mu+1}}(\Omega_1,\ldots,\Omega_p,Y,\phi_2,\ldots,\phi_u)\nabla_{i_2}\upsilon\ldots\nabla_{i_\mu}\upsilon$$

$$+\sum_{j\in J} a_j C_g^{j,i_2\ldots i_\mu}(\Omega_1,\ldots,\Omega_p,Y,\phi_2,\ldots,\phi_u)\nabla_{i_2}\upsilon\ldots\nabla_{i_\mu}\upsilon=0, \tag{4.28}$$

where $Op[C]_g^{r,i_2\ldots i_{\mu+1}}$ formally arises from by replacing the factor $S^\sharp\nabla^{(\rho)}_{r_1\ldots r_\rho}R_{iji_{\mu+1}l}\nabla^i\tilde{\phi}_q$ by $\nabla^{(\rho+2)}_{r_1\ldots r_\rho jl}Y\nabla_{i_{\mu+1}}\phi_q$ (and $Hit_Y\, div_{i_{\mu+1}}$ still means that $\nabla^{i_{\mu+1}}$ is forced to hit the factor $\nabla^{(B)}Y$). Then, formally replacing the expression

$$\nabla^{(C)}_{y_1\ldots y_C}X\nabla^{y_1}\upsilon\ldots\nabla^{y_a}\upsilon\nabla^{y_{a+1}}\phi_{w_1}\ldots\nabla^{y_b}\phi_{w_f}$$

by an expression

$$S^\sharp\nabla^{(C-2)}_{y_1\ldots y_{C-2}}R_{iy_{C-1}i_{\mu+1}y_C}\nabla^{y_1}\upsilon\ldots\nabla^{y_a}\upsilon\nabla^{y_{a+1}}\phi_{w_1}\ldots\nabla^{y_b}\phi_{w_f}\nabla^i\tilde{\phi}_q,$$

and repeating the formal identities by which (4.28) is proven "formally",[45] we derive (4.26). □

4.3. Derivation of Proposition 2.1 in case II from Lemma 3.2

Recall the notation of Lemma 3.2. Our point of departure will be the lemma's conclusion, Eq. (3.7). Recall that in that equation, all *tensor fields* have a given $(u+1)$-simple character, which we have denoted by $\vec{\kappa}'_{simp}$. We also recall that the $(\mu-1)$-tensor fields $\tilde{C}_g^{l,i_1\ldots i_\mu}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_h}\phi_{u+1}$ in the first line of (3.7) have *maximal refined double characters among all* $(\mu-1)$*-refined double characters appearing in* (3.7); we have denoted the maximal $(\mu-1)$-refined double characters by $\vec{\kappa}^z_{ref\text{-}doub}$, $z\in Z'_{Max}$.

For all tensor fields in (3.7) the factor $\nabla\phi_{u+1}$ is contracting against an index $_i$ in some chosen factor $S_*\nabla^{(\nu)}R_{ijkl}$. For this subsection, we will be calling that factor the A-crucial factor. We recall that all tensor fields in the first line in (3.7) have a given number of special free indices in the A-crucial factor. This follows from the definition of Z'_{Max}.

We further distinguish two subcases of case II: Observe that either all refined $(u+1,\mu-1)$-double characters $\vec{\kappa}^z$, $z\in Z'_{Max}$ one internal free index $_k$ or $_l$ in the A-crucial factor, or have no such free index.[46] We accordingly call these subcases A and B,[47] and we will prove our claim separately for these two subcases.

We here prove our assertion for all cases apart from certain *special cases* where we will derive the claim of Proposition 2.1 directly from (3.1) (in the paper [6] in this series).

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45 See the argument that proves (4.22).

46 Notice that this dichotomy corresponds to the following dichotomy regarding the tensor fields in $\bigcup_{z\in Z'_{Max}}L^z$ in (3.1): Either for those tensor fields the critical factor $\nabla^{(m)}R_{ijkl}$ contains two internal free indices, or it contains one internal free index.

47 Sometimes we will refer to subcases IIA, IIB, to stress that these are subcases of case II.

The special cases:

Case A: The special case here is when for each tensor field $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$, $z\in Z'_{Max}$ in (3.1) there are no removable free indices among any of its factors.[48]

Case B: The special case here is when for each tensor field $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$, $z\in Z'_{Max}$ in (3.1) there are no removable indices *other than* (possibly) the indices ${}_{k},{}_{l}$ in the (one of the) crucial factor $\nabla^{(m)}_{\upsilon_1\dots\upsilon_x i_1\dots i_b}R_{i_{b+1}jkl}$.[49]

Throughout the rest of this subsection we will be assuming that the μ-tensor fields of maximal refined double character in (3.1) are *not* special. In the special cases, Proposition 2.1 will be derived directly (without recourse to Lemma 3.2) in [6].

Derivation of Proposition 2.1 in case II from Lemma 3.2 (subcase A): Recall the conclusion of Lemma 3.2:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}X\,div_{i_1}\dots\widehat{X\,div}_{i_h}\dots X\,div_{i_\mu}\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\\
&\quad+\sum_{\nu\in N}a_\nu X\,div_{i_2}\dots X\,div_{i_\mu}C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&\quad+\sum_{d\in D}a_d X\,div_{i_1}\dots X\,div_{i_\mu}C^{d,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\\
&=\sum_{t\in T}a_t C^{t,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_u)\nabla_{i_*}\phi_{u+1}.
\end{aligned}\tag{4.29}$$

We now apply our inductive assumption of Corollary 1 to the above.[50] We derive that there is a linear combination of acceptable $(\mu-1)$-tensor fields with a $(u+1)$-simple character $\vec{\kappa}'_{simp}$ (indexed in H below), so that:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_h}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad+\sum_{\nu\in N}a_\nu C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad+\sum_{h\in H}a_h X\,div_{i_{\mu+1}}C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad+\sum_{j\in J}a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}\tag{4.30}$$

where each $C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ is simply subsequent to $Simp(\vec{\kappa}^z_{ref\text{-}doub})$.

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[48] Observe that if this is true of one of the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$, $z\in Z'_{Max}$, it will be true of all of them, by weight considerations.

[49] The remark in the previous footnote still holds.

[50] The fact that we are assuming that (2.3) (the assumption of Lemma 3.2) does not fall under a "special case" of subcase A ensures that (4.29) satisfies the requirements of Corollary 1. Also, since we have introduced a new factor $\nabla\phi_{u+1}$, the above falls under the inductive assumption of Corollary 1.

Note: We will now be calling the factor against which $\nabla\phi_{u+1}$ is contracting the A-crucial factor for *any* contraction appearing in the above.

Now, we observe that in view of the claim in Lemma 3.2, for each $\tilde{C}^{l,i_1\dots i_\mu}_g$, $C^{\nu,i_1\dots i_\mu}_g$ at least one of the indices $r_1,\dots,r_\nu,j$ in the A-crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ is not free and not contracting against a factor $\nabla\phi_f$, $f\leqslant u$. Furthermore, it follows from the definition of $\bigcup_{z\in Z'_{Max}}L^z$ that all $\tilde{C}^{l,i_1\dots i_\mu}_g$ tensor fields are contracting against a given number c of factors $\nabla\upsilon$. We denote by $\overline{N}\subset N$, $\overline{H}\subset H$, $\overline{J}\subset J$ the index sets of the contractions above for which the A-crucial factors is contracting against c factors $\nabla\upsilon$. Then, since (4.30) holds formally, we derive that:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_r\in I_{*,l}}\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_r}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_r}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\qquad+\sum_{\nu\in\overline{N}}a_\nu C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad=\sum_{h\in\overline{H}}a_h X\,div_{i_{\mu+1}}C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\qquad+\sum_{j\in\overline{J}}a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon.
\end{aligned}\tag{4.31}$$

In fact, we will be using a weaker version of this equation: Consider (4.31): We assume with no loss of generality, only for notational convenience, that in each $\tilde{C}^{l,i_1\dots i_\mu}$, $C^{\nu,i_1\dots i_\mu}_g$ and each $C^{h,i_1\dots i_{\mu+1}}$ the free indices $i_1,\dots,\hat{i_r},\dots,i_{c+1}$ belong to the A-crucial factor, while the indices $i_{c+2},\dots,i_\mu$ do not. Now, recall that (4.31) holds formally. Then, define an operation that formally erases the factors $\nabla\upsilon$ that are *not* contracting against the A-crucial factor and then takes $X\,div$s of the resulting free indices that we obtain. If we denote the expression that we (formally) thus obtain by F', it follows that $F'=0$ (modulo longer complete contractions) by virtue of the last lemma in the Appendix of [2]. We have thus derived[51]:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}X\,div_{i_{c+2}}\dots X\,div_{i_\mu}\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\qquad\cdot\nabla_{i_1}\upsilon\dots\nabla_{i_h}\phi_{u+1}\dots\nabla_{i_{c+1}}\upsilon\\
&\qquad+\sum_{\nu\in N}a_\nu X\,div_{i_{c+2}}\dots X\,div_{i_\mu}C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\\
&\quad=\sum_{h\in H}a_h X\,div_{i_{c+1}}\dots X\,div_{i_{\mu+1}}C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&\qquad\cdot\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon+\sum_{j\in J'}a_j C^{j,i_1\dots i_{c+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon.
\end{aligned}\tag{4.32}$$

Here the complete contractions indexed in J' are simply subsequent to $\vec{\kappa}'_{simp}$.

[51] We will revert to writing N, H, J instead of $\overline{N}$, $\overline{H}$, $\overline{J}$ for notational simplicity.

Now, we break the index set H into two subsets: $h \in H_1$ if and only if the A-crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ in

$$C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\pi}\upsilon$$

has the property that at least one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ is neither contracting against a factor $\nabla\upsilon$ nor a factor $\nabla\phi'_h$. We say that $h \in H_2$ if and only if all the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the A-crucial factor are contracting against a factor $\nabla\phi'_h$ or $\nabla\upsilon$. We complete our proof in two steps: Step 1 involves getting rid of the terms indexed in H_2:

Step 1: We introduce some notation:

Definition 4.3. We denote by $X_*\,div_i[\dots]$ the sublinear combination in each $X\,div_i[\dots]$ where we impose the additional restriction that ∇_i is not allowed to hit the A-crucial factor (in the form $S_*\nabla^{(\nu)}R_{ijkl}$).

Then, since (4.32) holds formally, we deduce that:

$$\sum_{h\in H_2} a_h X_*\,div_{i_{c+2}}\dots X_*\,div_{i_{\mu+1}} C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon$$
$$+\sum_{j\in J} a_j C_g^{j,i_2\dots i_{c+2}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+2}}\upsilon = 0 \tag{4.33}$$

(modulo longer terms), where each $C_g^{j,i_2\dots i_\pi}$ is simply subsequent to $\vec{\kappa}'_{simp}$.

A notational convention that can be made with no loss of generality is that in $\vec{\kappa}^z_{simp}$ the b factors $\nabla\phi'_h$ that are contracting against indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the A-crucial factor $S_*\nabla^{(\nu)}R_{ijkl}$ are precisely the factors $\nabla\phi_1,\dots,\nabla\phi_b$.

We then claim that we can write:

$$\sum_{h\in H_2} a_h X\,div_{i_{c+2}}\dots X\,div_{i_{\mu+1}} C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon$$
$$=\sum_{h\in \tilde{H}} a_h X\,div_{i_{c+2}}\dots X\,div_{i_{\mu+1}} C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon$$
$$+\sum_{j\in J} a_j C_g^{j,i_2\dots i_{c+2}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+2}}\upsilon, \tag{4.34}$$

where the sublinear combination indexed in $\tilde{H}$ in the RHS stands for a generic linear combination of tensor fields with all the properties of the tensor fields indexed in H_1 above.

We will show below that (4.34) follows by applying Lemma 4.6 or Lemma 4.7 to (4.33) (and we will prove Lemmas 4.6, 4.7 in [6]). For now, let us observe how (4.34) implies Proposition 2.1 in this case A:

Step 2: Proposition 2.1 follows from (4.34): By replacing (4.34) into (4.32), we are reduced to showing our claim when $H_2 = \emptyset$. Now, for each of the tensor fields in (4.32) we denote by $C_g^{l,i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$, $C_g^{\nu,i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)$,

$C_g^{h,i_{c+2}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$ the tensor fields that arise from $C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon$,

$$C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon$$

by replacing the expressions

$$S_*\nabla^{(A+c+b-1)}_{u_1\dots u_{A-1}s_1\dots s_b r_1\dots r_c}R_{iu_Akl}\nabla^{r_1}\upsilon\dots\nabla^{r_c}\upsilon\nabla^k\upsilon\nabla^{s_1}\phi_1\dots\nabla^{s_b}\phi_b\nabla^i\phi_{u+1}$$

by a factor $\nabla^{(A+1)}_{u_1\dots u_A l}Y$.

Now, by polarizing the function υ in (4.32) and applying Lemma 4.4, we deduce that:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}X\,div_{i_{c+2}}\dots X\,div_{i_\mu}\tilde{C}_g^{l,i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\\
&\quad+\sum_{\nu\in N}a_\nu X\,div_{i_{c+2}}\dots X\,div_{i_\mu}C_g^{\nu,i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\\
&=\sum_{h\in H}a_h X\,div_{i_{c+1}}\dots X\,div_{i_{\mu+1}}C_g^{h,i_{c+2}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&\quad+\sum_{j\in J}a_j C_g^j(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u).
\end{aligned}\tag{4.35}$$

Notice that all the tensor fields in the above have at least 2 derivatives on the factor $\nabla^{(b)}Y$. For the tensor fields indexed in H, this follows since $H_2=\emptyset$; for the tensor fields indexed in $(\bigcup_{z\in Z'_{Max}}L^z)\cup N$, it follows by our observation after (4.30). Thus, if we treat Y as a function Ω_{p+1}, all tensor fields in the above have a given simple character, which we will denote by $\tilde{\kappa}_{simp}$.

We denote the refined double character of the tensor fields $C_g^{l,i_{c+2}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$, $l\in L^z$, $z\in Z'_{Max}$ by $\vec{L}'_z$ (observe that they are the maximal refined double characters among all the $(\mu-c-1)$-tensor fields appearing in (4.35)). By virtue of our inductive assumption of Proposition 2.1,[52] we derive that for each $z\in Z'_{Max}$ there is a linear combination of acceptable $(\mu-c)$-tensor fields $\sum_{h\in H'}a_h C_g^{h,i_{c+1}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$ with an $(\mu-c-1)$-refined double character $\vec{L}'_z$ (so in particular they have a factor $\nabla^{(B)}Y$, $B\geqslant 2$, not contracting against any factor $\nabla\upsilon$, $\nabla\phi_h$), so that for each $z\in Z'_{Max}$:

$$\begin{aligned}
&\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}C_g^{l,i_1i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&\quad-\sum_{h\in H}a_h X\,div_{i_{\mu+1}}C_g^{h,i_{c+2}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon
\end{aligned}$$

52 Notice that *since we are assuming that the μ-tensor fields of maximal refined double character in* (3.1) *do not have special free indices in any factor* $S_*\nabla^{(\nu)}R_{ijkl}$ then it follows that the tensor fields of minimum rank in (4.35) satisfy the requirements of Proposition 2.1.

$$= \sum_{t\in T} a_t C^{t,i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon, \tag{4.36}$$

where each $C^{t,i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$ is acceptable and simply or doubly subsequent to $\vec{L}'_z$.

We use the fact that (4.36) holds formally. We then define an operation $Op[\dots]$ that replaces each factor $\nabla^{(B)}_{t_1\dots t_B}Y$ ($B\geqslant 2$) by an expression

$$\nabla^{(m)}_{s_1\dots s_b r_1\dots r_{c-1}t_1\dots t_{k-2}}R_{it_{B-1}kt_B}\nabla^i\upsilon\nabla^{r_1}\upsilon\dots\nabla^{r_{c-1}}\upsilon\nabla^k\upsilon\nabla^{s_1}\phi'_1\dots\nabla^{s_b}\phi'_b.$$

We observe that for each $z\in Z'_{Max}$:

$$Op\left\{\sum_{l\in L^z} a_l\sum_{i_h\in I_{*,l}}\tilde{C}^{l,i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon\right\}$$
$$= |I_{*,l}|\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon, \tag{4.37}$$

where we have noted that $|I_{*,l}|$ is universal, i.e. independent of the element $l\in L^z$ (in most cases $|I_{*,l}|=1$).

Hence, since (4.36) holds formally, we deduce that:

$$\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$-\sum_{h\in H} a_h X\,div_{i_{\mu+1}} Op\big[C^{h,i_{\pi+1}\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\nabla_{i_{\pi+1}}\upsilon\dots\nabla_{i_\mu}\upsilon\big]$$
$$= \sum_{t\in T'} a_t C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon, \tag{4.38}$$

where again each $C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ is either simply or doubly subsequent to $\vec{L}^z$. The above is obtained from (4.36) by the usual argument as in the derivation of Proposition 2.1 from Lemma 3.1 (see the argument above Eqs. (4.19), (4.20), (4.21)): We may repeat the permutations by which we make (4.36) formally zero, modulo introducing corrections terms that are simply or doubly subsequent by virtue of the Bianchi identities.

Therefore, we have shown that Lemma 3.2 implies case II of Proposition 2.1 in subcase A (in the non-special cases), provided we can prove (4.34). We reduce (4.38) to certain other lemmas, which will be proven in [6] in the next subsection.

Derivation of case II of Proposition 2.1 from Lemma 3.2 in case B: Our point of departure is again Eq. (3.7).

Recall that in this second case, for each $z\in Z'_{Max}$ *none* of the free indices in the A-crucial factor $S_*\nabla^{(\nu)}R_{ijkl}$ in any $\tilde{C}^{l,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$, $l\in L^z$ are special.

In that case, we again have Eq. (4.31). We will re-write the equation in a somewhat more convenient form, but first we recall some of the notational conventions from the previous case. For notational convenience, we have assumed that the b factors $\nabla\phi_h$, $h\leqslant u$ that are contracting against the A-crucial factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\phi_{u+1}$ are precisely the factors $\nabla\phi'_1,\dots,\nabla\phi'_b$. We also recall that $\vec{\kappa}^z_{ref\text{-}doub}$ stands for the $(u+1,\mu-1)$-refined double character of the contractions in $\sum_{i_h\in I_{*,l}}\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$. Eq. (4.31) can then be re-written in the form:

$$\begin{aligned}&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_r\in I_{*,l}}\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_r}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_r}\upsilon\dots\nabla_{i_\mu}\upsilon\\&\quad+\sum_{h\in H}a_h X\,div_{i_{\mu+1}}C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\&=\sum_{t\in T}a_t C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon,\end{aligned}\tag{4.39}$$

where each $C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ is (simply or doubly) subsequent to $\vec{\kappa}^z_{ref\text{-}doub}$. Moreover, if some $C_g^{t,i_1\dots i_\mu}$ is doubly subsequent to $\vec{\kappa}^z_{ref\text{-}doub}$ then at least one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the A-crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_n}R_{ijkl}$ is neither contracting against a factor $\nabla\upsilon$ nor against a factor $\nabla\phi_h$. The complete contractions on the right-hand side arise by indexing together the contractions in $\overline{N}$, $\overline{J}$ in (4.31).

We then again assume with no loss of generality that in each tensor field in the first line above, the indices ${}_{i_1},\dots,{}_{i_{c+1}}$ belong to the A-crucial factor and the indices ${}_{i_{c+2}},\dots,{}_{i_\mu}$ do not. We will then again use a weakened version of (4.39).

Weakened version of (4.39): Now, we return to (4.39). We derive an equation:

$$\begin{aligned}&\sum_{z\in Z'_{Max}}X\,div_{i_{c+2}}\dots X\,div_{i_\mu}\sum_{l\in L^z}a_l\sum_{i_r\in I_{*,l}}\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&\quad\cdot\nabla_{i_r}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_r}\upsilon\dots\nabla_{i_{c+1}}\upsilon\\&\quad+\sum_{t\in T'}a_t X\,div_{i_{c+2}}\dots X\,div_{i_\mu}C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\\&\quad+\sum_{h\in H}a_h X\,div_{i_{c+2}}\dots X\,div_{i_{\mu+1}}C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\\&\quad+\sum_{j\in J}a_j C_g^{j,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon=0;\end{aligned}\tag{4.40}$$

here each of the tensor fields indexed in T' are doubly subsequent to the maximal refined double characters $\vec{\kappa}^z_{ref\text{-}doub}$, while each of the complete contractions indexed in J is simply subsequent to $\vec{\kappa}'_{simp}$. This just follows from the previous equation by making the factors $\nabla\upsilon$ that are not contracting against the A-crucial factor into $X\,div$'s (as in the previous case A—we are applying the last lemma in the Appendix of [2] here).

Now, similarly to the previous case, we complete our proof in two steps; we first introduce some notation: We divide the index set H into two subsets: We say $h\in H_1$ if at least one of the indices ${}_{r_1},\dots,{}_{r_m},{}_j$ in the A-crucial factor $S_*\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$ does not contract against a factor $\nabla\upsilon$ or $\nabla\phi_h$, $h\leqslant u$. If all the indices ${}_{r_1},\dots,{}_{r_m},{}_j$ in the A-crucial factor $S_*\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$ contract against a factor $\nabla\upsilon$ or $\nabla\phi_f$, $f\leqslant u$, we say that $h\in H_2$. Now, step 1 involves getting rid of the terms indexed in H_2.

Step 1: For each $h \in H_2$, recall (from Definition 4.3) that $X_* div_i[\dots]$ is the sublinear combination in $X\, div_i[\dots]$, which arises under the extra restriction that the derivative ∇_i is not allowed to hit the A-crucial factor $S_*\nabla^{(m)}R_{ijkl}$. Then, since (4.39) holds formally, we deduce that:

$$\sum_{h\in H_2} a_h X_* div_{i_{c+2}}\dots X_* div_{i_\mu} X_* div_{i_{\mu+1}} C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$\cdot \nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon + \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) = 0. \tag{4.41}$$

We then claim that (4.41) will imply a new equation, for which we will need some more notation: Let us consider the tensor fields

$$\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_r}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_r}\upsilon\dots\nabla_{i_{c+1}}\upsilon,$$

$C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_r}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_r}\upsilon\dots\nabla_{i_{c+1}}\upsilon$ in (4.40).

For each $l \in L^z$, we denote by $\tilde{C}_g^{l,i_1 i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ the tensor field that arises from $\tilde{C}_g^{l,i_1\dots i_\mu}\nabla_{i_h}\phi_{u+1}$ (as it appears in (4.40)) by replacing the A-crucial factor

$$S_*\nabla^{(\nu)}_{i_2\dots i_{c+1}l_1\dots l_b y_{\pi+1}\dots y_\nu} R_{iy_{\nu+1}kl}\nabla^i\phi_{u+1}\nabla^{l_1}\phi_2\dots\nabla^{l_b}\phi_b\nabla^{i_2}\upsilon\dots\nabla^{i_{c+1}}\upsilon \tag{4.42}$$

($i_2,\dots,i_\pi$ are the free indices that belong to that A-crucial factor) by $S_*\nabla^{(\nu-\pi-b+1)}_{y_{\pi+1}\dots y_\nu}R_{iy_{\nu+1}kl}\cdot \nabla^i\phi_{u+1}$. We analogously define $C_g^{t,i_1 i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$. Notice these constructions are well-defined, since we know that at least one of the indices ${}_{i_2},\dots,{}_{r_{\nu+1}}$ in the left-hand side of (4.66) are not contracting against any factor $\nabla\phi$ or $\nabla\upsilon$.

Furthermore, observe that the tensor fields constructed above are acceptable, and have a given $(u-b)$-simple character, which we denote by $\vec{\kappa}''_{simp}$.

Now, our claim is that assuming (4.41), there is a linear combination of acceptable tensor fields (indexed in $\tilde{H}$ below) with a simple character $\vec{\kappa}''_{simp}$, and each with rank $\mu-c > \mu-c-1$ so that:

$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{i_r\in I_{*,l}} X\, div_{i_{c+2}}\dots X\, div_{i_\mu}\tilde{C}_g^{l,i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_r}\phi_{u+1}$$
$$+\sum_{t\in T'} a_t X\, div_{i_{c+2}}\dots X\, div_{i_\mu} C_g^{t,i_1 i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)$$
$$+\sum_{h\in\tilde{H}} a_h X\, div_{i_{c+2}}\dots X\, div_{i_{\mu+1}} C_g^{h,i_{c+2}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u) = 0; \tag{4.43}$$

here the contractions indexed in J are simply subsequent to $\vec{\kappa}''_{simp}$. The above holds modulo contractions of length $\geqslant \sigma+u-b+1$. This claim will be reduced to Lemma 4.8 in the next subsection. We now take it for granted and check how Proposition 2.1 in case IIB follows from (4.43).

Step 2: Derivation of Proposition 2.1 in case II subcase B from (4.43): Denote the refined double characters of the tensor fields in $\bigcup_{z\in Z'_{Max}} L^z$ by $\vec{\kappa}'^{z}_{ref\text{-}doub}$; observe that the tensor fields $C^{t,i_1 i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ are doubly subsequent to the refined double characters $\vec{\kappa}'^{z}_{ref\text{-}doub}$. Moreover, the refined double characters $\vec{\kappa}'^{z}_{ref\text{-}doub}$ are then all maximal.

Now, the above falls under the inductive assumption of Proposition 2.1[53]: If $b+c>0$ then the weight of the above complete contractions is $>-n$, and if $b+c=0$ then we have $u+1$ factors $\nabla\phi$. Thus we derive that for each $z\in Z'_{Max}$ there is a linear combination of acceptable tensor fields with a refined double character $\vec{\kappa}'^{z}_{ref\text{-}doub}$ (indexed in H^z below) so that:

$$\begin{aligned}
&\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} \tilde{C}^{l,i_h i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon \\
&\quad - X\,div_{i_{\mu+1}}\sum_{h\in H} a_h C^{h,i_1 i_{c+2}\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon \\
&= \sum_{t\in T} a_t C^{t,i_{c+1}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_{c+1}}\phi_{u+1}\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon
\end{aligned}\tag{4.44}$$

(here each $C^{t,i_{c+1}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_{c+1}}\phi_{u+1}$ is (simply or doubly) subsequent to $\vec{\kappa}'^{z}_{ref\text{-}doub}$).

Now, we define an operation $Add[\dots]$ that acts on the complete contractions and vector fields in the above by adding c derivative indices $\nabla_{g_1},\dots,\nabla_{g_c}$ to the A-crucial factor and then contracting them against c factors $\nabla\upsilon$, and then adds b derivative indices $\nabla_{f_1},\dots,\nabla_{f_b}$ onto the A-crucial factor and contracts them against factors $\nabla^{f_1}\phi_1,\dots,\nabla^{f_b}\phi_b$.

Since (4.44) holds formally, we derive that for each $z\in Z'_{Max}$:

$$\begin{aligned}
&\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} Add\big[C^{l,i_h i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon\big] \\
&\quad = \Big\{X\,div_{i_{\mu+1}}\sum_{h\in H} a_h Add\big[C^{h,i_{c+2}\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1} \\
&\qquad + \sum_{t\in T} a_t Add\big[C^{t,i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_{u+1})\big]\Big\}\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}\tag{4.45}$$

where each complete contraction indexed in T is simply or doubly subsequent to $\tilde{\kappa}^{z}_{ref\text{-}doub}$. If we set $\phi_{u+1}=\upsilon$ in the above, and we observe that:

$$\begin{aligned}
&\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} Add\big[\tilde{C}^{l,i_1 i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_h}\upsilon\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon\big] \\
&\quad = |I_{*,l}|\cdot\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon
\end{aligned}$$

[53] Observe that the tensor fields of minimum rank in (4.43) will not contain special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$ (since we are considering (3.1) in the setting of case IIB). Therefore there is no danger of falling under a "forbidden case" of Proposition 2.1.

$$+\sum_{t\in T''} a_t C_g^{t,i_1\dots i_\mu}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon, \tag{4.46}$$

where the complete contractions indexed in T'' are acceptable and doubly subsequent to $\vec{\kappa}^z_{ref\text{-}doub}$. We thus derive that the vector field needed for Proposition 2.1 in this case is precisely:

$$\frac{1}{|I_{*,l}|}\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} Add\big[\tilde{C}_g^{l,i_1 i_{\pi+1}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_h}\upsilon\nabla_{i_{c+2}}\upsilon\dots\nabla_{i_\mu}\upsilon\big].$$

Therefore we have shown that Lemma 3.2 implies Proposition 2.1 in case IIB, provided we can prove (4.43).

We now show how the claims (4.34) and (4.43) follow from four Lemmas 4.6, 4.8 and 4.7, 4.9, which we will state below. These four lemmas will be derived in the paper [6] in this series.

4.4. Reduction of the claims (4.34) and (4.43) to Lemmas 4.6, 4.8 and 4.7, 4.9 below

Reduction of claim (4.34) to Lemma 4.6: Since our Lemma 4.6 will also be used in other instances in this series of papers, we will re-write our hypothesis (Eq. (4.33)) in slightly more general notation:

We will set $c+1=\pi$ and write α instead of μ, to stress that our Lemma 4.6 below is independent of the specific values of the parameters μ, c. Furthermore, with no loss of generality, we will assume further down that $b=0$ (in other words that there are no factors $\nabla\phi'_h$ contracting against the crucial factor—this can be done since we can just re-name the factors $\nabla\phi'_h$ that contract against the A-crucial factor into $\nabla\upsilon$s). Now, recall the operation introduced in Step 2 after (4.34), where for each $h\in H_2$ we obtain tensor fields $C_g^{h,i_{\pi+1}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$ by formally replacing the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}\nabla^{r_1}\upsilon\dots\nabla^j\upsilon\nabla^i\tilde{\phi}_1\nabla^k\phi_{u+1}$ by an expression $\nabla_l Y$. As we noted after (4.33), if we apply this operation to a true equation, we again obtain a true equation. Thus, applying this operation to (4.33) we derive a new equation:

$$\begin{aligned}&\sum_{h\in H_2} a_h X_* div_{i_{\pi+1}}\dots X_* div_{i_{\mu+1}} C_g^{h,i_{\pi+1}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)\\ &\quad=\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u),\end{aligned} \tag{4.47}$$

where all complete contractions and tensor fields in the above have $\sigma+u-b-c$ factors, and are in the form:

$$\begin{aligned}&pcontr\big(\nabla^{(m_1)}R_{ijkl}\otimes\dots\otimes\nabla^{m_{\sigma_1}}R_{ijkl}\otimes S_*\nabla^{(\nu_1)}R_{ijkl}\otimes\dots\otimes S_*\nabla^{(\nu_t)}R_{ijkl}\otimes\nabla Y\\ &\quad\otimes\nabla^{(b_1)}\Omega_1\otimes\dots\otimes\nabla^{(b_p)}\Omega_p\otimes\nabla\phi_{z_1}\otimes\dots\otimes\nabla\phi_{z_w}\otimes\nabla\phi'_{z_{w+1}}\otimes\dots\otimes\nabla\phi'_{z_{w+d}}\\ &\quad\otimes\dots\otimes\nabla\tilde{\phi}_{z_{w+d+1}}\otimes\dots\otimes\nabla\tilde{\phi}_{z_{w+d+y}}\big).\end{aligned} \tag{4.48}$$

(Notice this is the same as the form (2.2), but for the fact that we have inserted a factor ∇Y in the second line.)

Definition 4.4. In the setting of (4.33) $X_* div_i$ will stand for the sublinear combination in $X div_i$ with the additional restriction that ∇_i is not allowed to hit the factor ∇Y. Moreover, we observe that the complete contractions in (4.47) have weight $-n+2(b+c)$.

Some language conventions before our next claim: We will be considering tensor fields $C^{i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)$ in the form (4.48), and even more generally in the form:

$$\begin{aligned}&pcontr\big(\nabla^{(m_1)}R_{ijkl}\otimes\cdots\otimes\nabla^{(m_{\sigma_1})}R_{ijkl}\otimes S_*\nabla^{(\nu_1)}R_{ijkl}\otimes\cdots\otimes S_*\nabla^{(\nu_t)}R_{ijkl}\otimes\nabla^{(B)}Y\\&\quad\otimes\nabla^{(b_1)}\Omega_1\otimes\cdots\otimes\nabla^{(b_p)}\Omega_p\otimes\nabla\phi_{z_1}\otimes\cdots\otimes\nabla\phi_{z_w}\otimes\nabla\phi'_{z_{w+1}}\otimes\cdots\otimes\nabla\phi'_{z_{w+d}}\otimes\cdots\\&\quad\otimes\nabla\tilde{\phi}_{z_{w+d+1}}\otimes\cdots\otimes\nabla\tilde{\phi}_{z_{w+d+y}}\big).\end{aligned}\tag{4.49}$$

(Notice this only differs from (4.48) by the fact that we allow $B\geqslant 1$ derivatives on the function Y.)

We will say that the tensor field in the form (4.49) is *acceptable* if all its factors are acceptable when we *disregard* the factor $\nabla^{(B)}Y$ (i.e. we may have $B=1$ derivatives on Y but the tensor fields will still be considered acceptable). Also, we will still use the notion of a *simple character* for such tensor fields (where we again just disregard the factor $\nabla^{(B)}Y$). With this convention, it follows that all the tensor fields in (4.35) have the same simple character, which we will denote by $\vec{\kappa}'_{simp}$. For such complete contractions σ will stand for the number of factors in one of the forms $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$, $\nabla^{(p)}\Omega_h$, $\nabla^{(B)}Y$.

We now state our technical lemma, which will be proven in the next paper in this series.

Lemma 4.6. *Assume an equation:*

$$\begin{aligned}&\sum_{h\in H_2}a_hX_*div_{i_1}\dots X_*div_{i_{a_h}}C^{h,i_1\dots i_{a_h}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})\\&\quad=\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u'}),\end{aligned}\tag{4.50}$$

where all tensor fields have rank $a_h\geqslant\alpha$. All tensor fields have a given u-simple character $\vec{\kappa}'_{simp}$, for which $\sigma\geqslant 4$. Moreover, we assume that if we formally treat the factor ∇Y as a factor $\nabla\phi_{u'+1}$ in the above equation, then the inductive assumption of Proposition 2.1 *can be applied.*

The conclusion (under various assumptions which we will explain below): Denote by $H_{2,\alpha}$ the index set of tensor fields with rank α.

We claim that there is a linear combination of acceptable[54] *tensor fields, $\sum_{d\in D}a_d\cdot C^{d,i_1\dots i_{\alpha+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)$, each with a simple character $\vec{\kappa}'_{simp}$ so that:*

$$\begin{aligned}&\sum_{h\in H_{2,\alpha}}a_hC^{h,i_1\dots i_\alpha}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\upsilon\\&\quad-X_*div_{i_{\alpha+1}}\sum_{d\in D}a_dC^{d,i_1\dots i_{\alpha+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\upsilon\end{aligned}$$

[54] "Acceptable" in the sense given after (4.49).

$$=\sum_{t\in T} a_t C^t_g\big(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'},\upsilon^\alpha\big). \tag{4.51}$$

The linear combination on the right-hand side stands for a generic linear combination of complete contractions in the form (4.48) *with a factor* ∇Y *and with a simple character that is subsequent to* $\vec{\kappa}'_{simp}$.

The assumption under which (4.52) *holds is that there should be no tensor fields of rank* α *in* (4.50) *which are "bad". Here "bad" means the following:*

If $\sigma_2=0$ *in* $\vec{\kappa}'_{simp}$ *then a tensor field in the form* (4.48) *is "bad" provided:*

1. *The factor* ∇Y *contains a free index.*
2. *If we formally erase the factor* ∇Y *(which contains a free index), then the resulting tensor field should have no removable indices,*[55] *and no free indices.*[56]

If $\sigma_2>0$ *in* $\vec{\kappa}'_{simp}$ *then a tensor field in the form* (4.48) *is "bad" provided:*

1. *The factor* ∇Y *should contain a free index.*
2. *If we formally erase the factor* ∇Y *(which contains a free index), then the resulting tensor field should have no removable indices, any factors* S_*R_{ijkl} *should be simple, any factor* $\nabla^{(2)}_{ab}\Omega_h$ *should have at most one of the indices* $_a$, $_b$ *free or contracting against a factor* $\nabla\phi_s$.
3. *Any factor* $\nabla^{(m)}R_{ijkl}$ *can contain at most one (necessarily special, by virtue of* 2.*) free index.*

Furthermore, we claim that the proof of this lemma will only rely on the inductive assumption of Proposition 2.1. *Moreover, we claim that if all the tensor fields indexed in* H_2 *(in* (4.50)*) do not have a free index in* ∇Y *then we may assume that the tensor fields indexed in* D *in* (4.52) *have the same property.*

Note: It follows (by weight considerations) that none of the tensor fields of minimum rank in (4.47) is "bad" in the above sense, since our assumption (3.1) *does not* fall under one of the special cases, as described in the beginning of this subsection.

We also claim a corollary of Lemma 4.6. Firstly, we introduce some notation:

$$\sum_{q\in Q} a_q C^{q,i_{\pi+1}\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_u)$$

will stand for a generic linear combination of acceptable tensor fields with a simple character $\vec{\kappa}'_{simp}$ and with a factor $\nabla^{(B)}Y$ with $B\geqslant 2$ (and where this factor is not contracting against any factors $\nabla\phi_h$).

Corollary 2. *Under the assumptions of Lemma* 4.6*, with* $\sigma\geqslant 4$ *we can write:*

[55] Thus, the tensor field should consist of factors S_*R_{ijkl}, $\nabla^{(2)}\Omega_h$, and factors $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$ with all the indices $r_1,\dots,r_m$ contracting against factors $\nabla\phi_h$.

[56] I.e. $\alpha=1$ in (4.50).

$$\sum_{h\in H_2} a_h X\,div_{i_1}\dots X\,div_{i_\alpha} C_g^{h,i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$=\sum_{q\in Q} a_q X\,div_{i_1}\dots X\,div_{i_{a'}} C_g^{q,i_1\dots i_{a'}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$+\sum_{t\in T} a_t C_g^t(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'}), \tag{4.52}$$

where the linear combination $\sum_{q\in Q} a_q C_g^{q,i_1\dots i_{a'}}$ *stands for a generic linear combination of tensor fields in the form* (4.49) *with* $B\geqslant 2$*, with a simple character* $\vec{\kappa}'_{simp}$ *and with each* $a'\geqslant\alpha$*. The acceptable complete contractions* $C_g^t(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$ *are simply subsequent to* $\vec{\kappa}'_{simp}$*.* $X\,div_i$ *here means that* ∇_i *is not allowed to hit the factors* $\nabla\phi_h$ *(but it is allowed to hit* $\nabla^{(B)}Y$*).*

We have an analogue of the above corollary when $\sigma=3$ (the next Lemma 4.7, will also be proven in the paper [6]).

Lemma 4.7. *We assume* (4.50)*, where* $\sigma=3$*. We also assume that for each of the tensor fields in* $H_2^{\alpha,*}$[57] *there is at least one removable index. We then have two claims:*

Firstly, the conclusion of Lemma 4.6 *holds in this setting also. Secondly, the conclusion of Corollary* 2 *is true in this setting.*

Before we show that Corollary 2 follows from Lemma 4.6, let us see how our desired equation (4.34) follows from the above corollary:

Corollary 2 (or Lemma 4.7 when $\sigma=3$*) implies (4.34):* We introduce an operation $Op\{\dots\}$ which acts on complete contractions and tensor fields in the form (4.49) by formally replacing the factor $\nabla^{(B)}_{r_1\dots r_B}Y$ (recall $B\geqslant 1$) by

$$S_*\nabla^{(B+b+c-2)}_{y_1\dots y_b s_1\dots s_c r_1\dots r_{B-2}} R_{i r_{B-1} s_{c+1} r_B}\nabla^{y_1}\phi_1\dots\nabla^{y_b}\phi_b\nabla^{s_1}\upsilon\dots\nabla^{s_{c+1}}\upsilon.$$

Then, if we apply this operation to (4.52) and we repeat the permutations by which we make (4.52) formally zero (modulo introducing correction terms by the Bianchi identities (4.19), (4.20), (4.21)), we derive (4.34). So matters are reduced to showing that Corollary 2 follows from Lemma 4.6. □

Proof that Corollary 2 follows from Lemma 4.6. The proof is by induction. Firstly, we apply Lemma 4.6 and we pick out the sublinear combination in the conclusion of Lemma 4.6 where ∇Y is contracting against a factor $\nabla\upsilon$. That sublinear combination must vanish separately, thus we obtain an equation:

$$\sum_{h\in H_2^{\alpha,*}} a_h C_g^{h,i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\upsilon$$

[57] Recall that $H_2^{\alpha,*}$ is the index set of tensor fields of rank α in (4.50) with a free index in the factor ∇Y.

$$- X_* div_{i_{\alpha+1}} \sum_{d\in D'} a_d C_g^{d,i_1\dots i_{\alpha+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\upsilon$$
$$= \sum_{t\in T} a_t C_g^t\big(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'},\upsilon^\alpha\big). \tag{4.53}$$

Now, we make the factors $\nabla\upsilon$ into $X\,div$s[58] (which *are* allowed to hit the factor ∇Y) and we derive a new equation:

$$\sum_{h\in H_2^{\alpha,*}} a_h X\,div_{i_1}\dots X\,div_{i_\alpha} C_g^{h,i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$- X\,div_{i_1}\dots X\,div_{i_\alpha} X\,div_{i_{\alpha+1}} \sum_{d\in D'} a_d C_g^{d,i_1\dots i_{\alpha+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$= \sum_{q\in Q} a_q X\,div_{i_1}\dots X\,div_{i_\alpha} C_g^{q,i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$+ \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'}). \tag{4.54}$$

In view of this equation, we are reduced to proving our claim when $H_2^{\alpha,*}=\emptyset$. That is, we may then additionally assume that in the hypothesis of Lemma 4.6 no tensor fields contain a free index in ∇Y (if there are such tensor fields with a factor $\nabla_{i_1}Y$, we just treat $X_* div_{i_1}\nabla_{i_1}Y[\dots]$ as a sum of β-tensor fields, $\beta\geqslant\alpha$). We will be making this assumption until the end of this proof.

Then we proceed by induction. More precisely, our inductive statement is the following: Suppose we know that for some number $f\geqslant 0$ we can write:

$$\sum_{h\in H_2} a_h X\,div_{i_1}\dots X\,div_{i_a} C_g^{h,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$= \sum_{q\in Q} a_q X\,div_{i_1}\dots X\,div_{i_a} C_g^{q,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$+ \sum_{h\in H_2^f} a_h X\,div_{i_1}\dots X\,div_{i_{a+f}} C_g^{h,i_1\dots i_{a+f}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$
$$+ \sum_{t\in T} a_t C_g^t(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'}), \tag{4.55}$$

where the tensor fields indexed in H_2^f still have a factor ∇Y (which *does not* contain a free index) but are otherwise acceptable with simple character $\vec{\kappa}'_{simp}$ and have rank $\alpha+f$.

Our claim is that we can then write:

$$\sum_{h\in H_2} a_h X\,div_{i_1}\dots X\,div_{i_a} C_g^{h,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'})$$

58 See the last lemma in the Appendix of [2].

$$= \sum_{q\in Q} a_q X\,div_{i_1}\ldots X\,div_{i_a} C_g^{q,i_1\ldots i_a}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})$$
$$+ \sum_{h\in H_2^{f+1}} a_h X\,div_{i_1}\ldots X\,div_{i_{a+f+1}} C_g^{h,i_1\ldots i_{a+f+1}}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})$$
$$+ \sum_{t\in T} a_t C_g^t(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'}) \tag{4.56}$$

(with the same convention regarding $\sum_{h\in H_2^{f+1}}\ldots$—it is like the sublinear combination $\sum_{h\in H_2^f}\ldots$ only with rank $\geqslant f+1$).

Clearly, if we can show this inductive step then our corollary will follow, since we are dealing with tensor fields of a *fixed* weight $-K$, $K\leqslant n$.

This inductive step is not hard to deduce. Assuming (4.55), we pick out the sublinear combination that contains a factor ∇Y (which vanishes separately) and we replace it into (4.47) to derive the equation:

$$\sum_{h\in H_2^f} a_h X_*\,div_{i_1}\ldots X_*\,div_{i_{\alpha+f}} C_g^{h,i_1\ldots i_{\alpha+f}}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})$$
$$= \sum_{t\in T} a_t C_g^t(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'}). \tag{4.57}$$

Now, applying Lemma 4.6 to this equation,[59] we derive that there is a linear combination of acceptable $(\alpha+f+1)$-tensor fields (indexed in D^f below) with a factor ∇Y and a simple character $\vec{\kappa}'_{simp}$ so that:

$$\sum_{h\in H_2^f} a_h C_g^{h,i_1\ldots i_{\alpha+f}}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})\nabla_{i_1}\upsilon\ldots\nabla_{i_{\alpha+f+1}}\upsilon$$
$$- X_*\,div_{i_{\alpha+f+1}} \sum_{d\in D^f} a_d C_g^{d,i_1\ldots i_{\alpha+f+1}}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})\nabla_{i_1}\upsilon\ldots\nabla_{i_{\alpha+f}}\upsilon$$
$$= \sum_{t\in T} a_t C_g^t\big(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'},\upsilon^{\alpha-\pi}\big). \tag{4.58}$$

But observe that the above implies:

$$\sum_{h\in H_2^f} a_h C_g^{h,i_1\ldots i_{\alpha+f}}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})\nabla_{i_1}\upsilon\ldots\nabla_{i_{\alpha+f}}\upsilon$$
$$- X\,div_{i_{\alpha+f+1}} \sum_{d\in D^f} a_d C_g^{d,i_1\ldots i_{\alpha+f+1}}(\Omega_1,\ldots,\Omega_p,Y,\phi_1,\ldots,\phi_{u'})\nabla_{i_1}\upsilon\ldots\nabla_{i_{\alpha+f}}\upsilon$$

[59] Since we are assuming that the terms of maximal refined double character in the hypothesis of Proposition 2.1 are assumed not to be "special" (as defined in the beginning of the previous subsection), it follows by weight considerations that no terms in (4.57) are "bad" in the language of Lemmas 4.6.

$$= \sum_{q\in Q} a_q C_g^{q,i_1\dots i_{\alpha+f}}(\Omega_1,\dots,\Omega_p,Y,\phi_{b+1},\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_{\alpha+f}}\upsilon$$

$$+\sum_{t\in T} a_t C_g^t\big(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_{u'},\upsilon^{\alpha+f+1}\big), \tag{4.59}$$

where the tensor fields indexed in Q are acceptable with a simple character $\vec{\kappa}'_{simp}$ and with a factor $\nabla^{(2)}Y$. But then just making the $\nabla\upsilon$'s into $X\,div$'s in the above we obtain (4.56) with $H^{f+1}=D^f$. $\square$

Reduction of Eq. (4.43) to Lemmas 4.8, 4.9 below: We define an operation that acts on the tensor fields in (4.39) and (4.41) by replacing the expression

$$S_*\nabla^{(\nu+b)}_{s_1\dots s_b r_1\dots r_\nu}R_{ijkl}\nabla^i\phi_{u+1}\nabla^{r_1}\upsilon\dots\nabla^{r_\tau}\upsilon\nabla^{s_1}\phi'_1\dots\nabla^{s_b}\phi'_b$$

by an expression $\nabla_{(r_{\tau+1}\dots r_m j)l}\omega_1\nabla_k\omega_2-\nabla_{(r_{\tau+1}\dots r_m j)k}\omega_1\nabla_l\omega_2$. We denote this operation by *Repl*$\{\dots\}$. Thus, acting with the above operation we obtain complete contractions and tensor fields in the form:

$$contr\big(\nabla^{(m_1)}R_{ijkl}\otimes\dots\otimes\nabla^{(m_s)}R_{ijkl}\otimes S_*\nabla^{(\nu_1)}R_{ijkl}\otimes\dots\otimes S_*\nabla^{(\nu_b)}R_{ijkl}$$
$$\otimes\nabla^{(B,+)}_{r_1\dots r_B}(\nabla_a\omega_1\nabla_b\omega_2-\nabla_b\omega_1\nabla_a\omega_1)\otimes\nabla^{(d_1)}\Omega_p\otimes\dots\otimes\nabla^{(d_p)}\Omega_p\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_u\big); \tag{4.60}$$

here $\nabla^{(B,+)}_{r_1\dots r_B}(\dots)$ stands for the sublinear combination in $\nabla^{(B)}_{r_1\dots r_B}(\dots)$ where each ∇ is not allowed to hit the factor $\nabla\omega_2$.

Definition 4.5. We define the simple character of a complete contraction or tensor field in the above form to be the simple character of the complete contraction or tensor fields that arises from it by *disregarding* the two factors $\nabla^{(B)}\omega_1$, $\nabla\omega_2$. For each tensor field in the form (4.60), we will also define σ to stand for the number of factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$, $\nabla^{(p)}\Omega_h$ *plus one*. (In other words, we are not counting the $\nabla\phi$'s and we are counting the two factors ω_1,ω_2 as one.)

We then derive from (4.40) that:

$$\sum_{z\in Z'_{Max}} X^+div_{i_{c+2}}\dots X^+div_{i_\mu}$$
$$\cdot\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}Repl\big\{C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_1}\upsilon\dots\hat{\nabla}_{i_h}\upsilon\dots\nabla_{i_{c+1}}\upsilon\big\}$$
$$+\sum_{t\in T'}a_t X^+div_{i_{c+2}}\dots X^+div_{i_\mu}Repl\big\{C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$\cdot\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+2}}\upsilon\big\}-X^+div_{i_{c+2}}\dots X^+div_{i_{\mu+1}}$$
$$\cdot\sum_{h\in H}a_h Repl\big\{C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\big\}$$

$$= \sum_{j \in J} a_j \, Repl\big\{C_g^{j,i_1\dots i_{c+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\big\}, \tag{4.61}$$

where here $X^+ \, div_i$ stands for the sublinear combination in $X \, div_i$ where ∇_i is not allowed to hit the factor $\nabla\omega_2$. This equation follows from the *proof* of the last lemma in the Appendix in [2].[60] Thus, we derive that:

$$\sum_{h \in H_2} a_h X_* \, div_{i_{c+2}} \dots X_* \, div_{i_\mu} X_* \, div_{i_{\mu+1}}$$
$$\cdot Repl\big\{C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\big\}$$
$$= \sum_{j \in J'} a_j \, Repl\big\{C_g^{t,i_1\dots i_{c+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_{c+1}}\upsilon\big\}, \tag{4.62}$$

where here X_* stands for the sublinear combination in $X \, div_i$ where ∇_i is not allowed to hit either of the factors $\nabla\omega_1, \nabla\omega_2$. Also $J' \subset J$ stands for the sublinear combination of complete contraction with two factors $\nabla\omega_1, \nabla\omega_2$ (each with only one derivative).

We next formulate Lemma 4.8, which we will show in [6]. We introduce one further piece of notation before stating this claim:

Definition 4.6. Let $C_g^{x,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_{u'})$ stand for a tensor field in the form (4.60) with $B = 0$. We will say that a derivative index in some factor $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$ in $C_g^{x,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_{u'})$ is "removable" if it is neither free not contracting against a factor $\nabla\phi_h$.

Now, consider any factor $\nabla^{(B)}_{r_1\dots r_B}\Omega_v$ in $C_g^{x,i_1\dots i_a}$, where we make the normalizing requirement that all indices that are either free or are contracting against a factor $\nabla\phi_h$ or $\nabla\omega_f$ are pulled to the right. We then say that an index in $\nabla^{(B)}_{r_1\dots r_B}\Omega_v$ is "removable" if it is one of the leftmost $B-2$ indices and it is neither free, nor contracting against any factor $\nabla\phi_h, \nabla\omega_f$.

Lemma 4.8. *Consider a linear combination of partial contractions,*

$$\sum_{x \in X} a_x C_g^{x,i_1\dots i_a}\big(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_{u'}\big),$$

where each of the tensor fields $C_g^{x,i_1\dots i_a}$ is in the form (4.60) *with $B = 0$ (and is antisymmetric in the factors $\nabla_a\omega_1, \nabla_b\omega_2$ by definition), with rank $a \geqslant \alpha$ and length $\sigma \geqslant 4$.*[61] *We assume all these tensor fields have a given simple character which we denote by $\vec{\kappa}'_{simp}$ (we use u' instead of u to stress that this lemma holds in generality). We assume an equation:*

$$\sum_{x \in X} a_x X_* \, div_{i_1} \dots X_* \, div_{i_a} C_g^{x,i_1\dots i_a}\big(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_u\big)$$

60 By repeating exactly the same argument.

61 Recall we are counting the two factors ω_1, ω_2 for one.

$$+\sum_{j\in J} a_j C^j_g\big(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_u\big)=0, \tag{4.63}$$

where $X_* div_i$ stands for the sublinear combination in $X div_i$ where ∇_i is in addition not allowed to hit the factors $\nabla\omega_1,\nabla\omega_2$. The contractions C^j here are simply subsequent to $\vec{\kappa}'_{simp}$. We assume that if we formally treat the factors $\nabla\omega_1,\nabla\omega_2$ as factors $\nabla\phi_{u+1},\nabla\phi_{u+2}$ (disregarding whether they are contracting against special indices) in the above, then the inductive assumption of Proposition 2.1 *applies.*

The conclusion we will draw (under various hypotheses that we will explain below) is that we can write:

$$\sum_{x\in X} a_x X_+ div_{i_1}\dots X_+ div_{i_a} C^{x,i_1\dots i_a}_g\big(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_u\big)$$
$$=\sum_{x\in X'} a_x X_+ div_{i_1}\dots X_+ div_{i_a} C^{x,i_1\dots i_a}_g\big(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_u\big)$$
$$+\sum_{j\in J} a_j C^j_g\big(\Omega_1,\dots,\Omega_p,[\omega_1,\omega_2],\phi_1,\dots,\phi_u\big), \tag{4.64}$$

where the tensor fields indexed in X' on the right-hand side are in the form (4.60) *with $B>0$. All the other sublinear combinations are as above. We recall that $X_+ div_i$ stands for the sublinear combination in $X div_i$ where ∇_i is in addition not allowed to hit the factor $\nabla\omega_2$ (it is allowed to hit the factor $\nabla^{(B)}\omega_1$).*

Assumptions needed to derive (4.64): We claim (4.64) *under certain assumptions on the α-tensor fields in* (4.63) *which have rank α and have a free index in one of the factors $\nabla\omega_1,\nabla\omega_2$ (say to $\nabla\omega_1$ wlog)—we denote the index set of those tensor fields by $X^{\alpha,*}\subset X$.*

The assumption we need in order for the claim to hold is that no tensor field indexed in $X^{\alpha,*}$ should be "bad". A tensor field is "bad" if it has the property that when we erase the expression $\nabla_{[a}\omega_1\nabla_{b]}\omega_2$ (and make the index that contracted against $_b$ into a free index) then the resulting tensor field will have no removable indices, and all factors S_*R_{ijkl} will be simple.

Lemma 4.9. *We assume* (4.63)*, where now the tensor fields have length $\sigma=3$. We also assume that for each of the tensor fields indexed in X, there is a removable index in each of the real factors. We then claim that the conclusion of Lemma* 4.8 *is still true in this setting.*

We will show Lemmas 4.8, 4.9 in [6]. For now, let us see how they imply (4.43).

Note: Observe that (4.62) satisfies the requirements of Lemma 4.8 by weight considerations, since we are assuming that (3.1) *does not* fall under any of the "special cases" outlined in the beginning of the previous subsection.

Thus, we will now apply Lemma 4.8 (or 4.9) to (4.62).

Consider (4.62). We denote by $\vec{\kappa}_*$ the simple character of the tensor fields $Repl\{C^{h,i_1\dots i_{\alpha+1}}_g\}$. We then observe that Lemmas 4.8 (or 4.9) imply:

$$\sum_{h\in H_2} a_h X^+ div_{i_{c+2}} X^+ div_{i_{\mu+1}} Repl\big\{C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\pi}\upsilon\big\}$$

$$= \sum_{q\in Q} a_q X^+ div_{i_{c+2}} X^+ div_{i_\beta} C_g^{q,i_{c+2}\dots i_\beta}(\Omega_1,\dots,\Omega_p,\omega_1,\omega_2,\phi_1,\dots,\phi_u)$$
$$+ \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\omega_1,\omega_2,\phi_1,\dots,\phi_u), \tag{4.65}$$

where each $C_g^{q,i_{c+2}\dots i_\beta}(\Omega_1,\dots,\Omega_p,\omega_1,\omega_2,\phi_1,\dots,\phi_u)$ $(\beta \geqslant \mu+1)$ is a generic acceptable tensor field in the form (4.60), with the additional restriction that it has an expression $\nabla^+_\chi(\nabla_a\omega_1\nabla_b\omega_2 - \nabla_a\omega_2\nabla_b\omega_1)$.[62] Also, each $C_g^j(\Omega_1,\dots,\Omega_p,\omega_1,\omega_2,\phi_1,\dots,\phi_u)$ is in the form (4.60) with $(B=0)$ but is also simply subsequent to $\vec{\kappa}_*$.

We now define an operation Op_* which formally acts on the complete contractions (and linear combinations thereof) in (4.61) by replacing each expression $\nabla^{(K)}_{(r_1\dots r_K)}\omega_1\nabla_\gamma\omega_2$[63] by an expression:

$$(K-1)\cdot\nabla^{(K+1)}_{\gamma(r_1\dots r_K)}\phi_{u+1}.$$

This operation can also be defined on the tensor fields appearing in (4.40). Before we proceed to explain how this operation can act on true equations and produce true equations, let us see what will be the outcome of formally applying Op_* to Eq. (4.61):

Op_* acting on (4.61) proves (4.43): For each $l\in L^z$, we denote by

$$\tilde{C}_g^{l,i_1 i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$

the tensor field that arises from $\tilde{C}_g^{l,i_1\dots i_\mu}\nabla_{i_h}\phi_{u+1}$ (as it appears in (4.40)) by replacing the A-crucial factor

$$S_*\nabla^{(\nu)}_{i_2\dots i_{c+1}l_1\dots l_b y_f\dots y_\nu} R_{i r_{\nu+1} kl}\nabla^i\phi_{u+1}\nabla^{l_1}\phi_2\dots\nabla^{l_b}\phi_b\nabla^{i_2}\upsilon\dots\nabla^{i_{c+1}}\upsilon \tag{4.66}$$

$({}_{i_2},\dots,{}_{i_\pi}$ are the free indices that belong to that critical factor) by $S_*\nabla^{(\nu-\pi-b+1)}_{y_{\pi+1}\dots y_\nu}R_{ijkl}\nabla^i\phi_{u+1}$.

We analogously define the tensor fields $C_g^{h,i_1 i_{c+2}\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)$, $C_g^{t,i_1 i_{c+2}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$. Observe that for the tensor fields indexed in H, this is a well-defined operation, since we are assuming that $H_2=\emptyset$ in (4.61) (thus for each of the tensor fields above we will have that at least one of the indices ${}_{i_1},\dots,{}_{r_{\nu+1}}$ in (4.66) is not contracting against a factor $\nabla\phi$ or $\nabla\upsilon$).

We observe that for each $l\in L^z$, $z\in Z'_{Max}$:

$$Op_*\big\{X\,div_{i_{\pi+1}}\dots X\,div_{i_\mu} Repl\big\{\tilde{C}_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\pi}\upsilon\big\}\big\}$$
$$= X\,div_{i_{\pi+1}}\dots X\,div_{i_\mu} C_g^{l,i_1 i_{\pi+1}\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+ \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_{u+1}) \tag{4.67}$$

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[62] Recall that ∇^+_χ stands for the sublinear combination in ∇_χ where ∇_χ is not allowed to hit $\nabla\omega_2$.

[63] ${}_{(\dots)}$ stands for symmetrization of the indices between parentheses.

(the tensor fields and complete contractions on the right-hand side have length $\sigma - b + u$. Here each C^j_g has length $\sigma - b + u + 1$) and also has a factor $\nabla^{(s)}\phi_{u+1}$, $s > 1$.

In the same way we derive that for each $h \in H$ (recall that $H_2 = \emptyset$, hence the factor $\nabla^{(K)}\omega_1$ in each $C^{h,i_1\dots i_{\mu+1}}_g$ has $K > 1$) and for each $t \in T'$:

$$\begin{aligned} &Op_*\big\{X\,div_{i_{c+2}}\dots X\,div_{i_{\mu+1}}\,Repl\big\{C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\pi}\upsilon\big\}\big\} \\ &\quad = X\,div_{i_{c+2}}\dots X\,div_{i_{\mu+1}}C^{h,i_1 i_{c+2}\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u) \\ &\qquad + \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u), \end{aligned} \tag{4.68}$$

$$\begin{aligned} &Op_*\big\{X\,div_{i_{c+2}}\dots X\,div_{i_\mu}\,Repl\big\{C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\pi}\upsilon\big\}\big\} \\ &\quad = X\,div_{i_{c+2}}\dots X\,div_{i_\mu}C^{t,i_1 i_{c+2}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_u)\nabla_{i_1}\phi_{u+1} \\ &\qquad + \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{b+1},\dots,\phi_{u+1}), \end{aligned} \tag{4.69}$$

where each C^j_g has length $\sigma - b + u + 1$ and also has a factor $\nabla^{(s)}\phi_{u+1}$, $s > 1$.

Op_* produces a true equation: Now, let us explain why acting on (4.61) by Op_* will produce a true equation: We break up (4.61) (denote its left-hand side by F) into sublinear combinations F^K, according to the number K of derivatives on the factor $\nabla^{(K)}\omega_1$. Since (4.61) holds formally, it follows that $F^K = 0$ formally (modulo longer contractions). We then apply Op_* to each equation $F^K = 0$. This produces a true equation since we may just repeat the permutations by which F^K is made formally zero to $Op_*\{F^K\}$. Adding over all equations $Op_*\{F^K\} = 0$, we derive our conclusion.

For future reference, we formulate a corollary of Lemma 4.8:

Corollary 3. *We consider a linear combination of α-tensor fields of weight $-n+\alpha$ and length σ*

$$\sum_{w\in W} a_w C^{w,i_1\dots i_\alpha}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b),$$

where each tensor field above has a given b-simple character $\vec{\kappa}_*$ and a given rank α. We assume that for a given factor $T = S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ (for which the index $_i$ is contracting against a given factor $\nabla\tilde{\phi}_k$) each tensor field indexed in W has the feature that the factor T has at least one of the indices $_{r_1},\dots,_{r_\nu},_j$ not contracting against a factor $\nabla\phi$ (for this lemma only we refer to this as the good property). Assume an equation of the form:

$$\begin{aligned} &\sum_{w\in W} a_w X\,div_{i_1}\dots X\,div_{i_\alpha}C^{w,i_1\dots i_\alpha}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b) \\ &\quad + \sum_{h\in H} a_h X\,div_{i_1}\dots X\,div_{i_z}C^{h,i_1\dots i_z}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_b) \end{aligned}$$

$$= \sum_{j \in J'} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_b), \tag{4.70}$$

where each tensor field indexed in H has a b-simple character $\vec{\kappa}_*$ has rank $z \geqslant \beta$ (for some chosen β), and does not satisfy the good property. Furthermore, we assume that for these tensor fields of rank exactly β, if we formally replace the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu} R_{ijkl}$ by $\nabla^{(\nu+1)}_{r_1\dots r_\nu j[k}\omega_1 \nabla_{l]}\omega_2$, then the resulting tensor fields satisfy the hypotheses of Lemma 4.8 *or* 4.9. *Each complete contraction indexed in J' is simply subsequent to $\vec{\kappa}_*$.*

We then claim that:

$$\sum_{w \in W} a_w X\,div_{i_1} \dots X\,div_{i_\alpha} C^{w,i_1\dots i_\alpha}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_b)$$
$$+ \sum_{h \in H'} a_h X\,div_{i_1} \dots X\,div_{i_z} C^{h,i_1\dots i_z}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_b)$$
$$= \sum_{j \in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_b), \tag{4.71}$$

where each tensor field indexed in H' are as the tensor field indexed in H above, only they now satisfy the good property. Each complete contraction indexed in J is simply subsequent to $\vec{\kappa}_*$.

Proof. The proof just follows by reiterating the argument above: We first use the operation *Repl*$\{\dots\}$ as above, and then apply Lemma 4.8. We then use the operation *Op*$\{\dots\}$ as above and then the operation *Add*. □

4.5. Derivation of Proposition 2.1 from Lemma 3.5

Firstly observe that we only have to show the above claim in case A, since in case B the claims of Lemma 3.5 and Proposition 2.1 coincide.

Proof in two steps. We show that Lemma 3.5 (in case A) implies Proposition 2.1 in steps: Firstly, using the conclusion of Lemma 3.5 we show that we can derive a new equation:

$$\binom{\alpha}{2} \sum_{z \in Z'_{Max}} \sum_{l \in L^z} a_l \sum_{r=0}^{k-1} X\,div_{i_2} \dots X\,div_{i_*} \dot{C}^{l,i_1\dots\hat{i}_{r\alpha+1}\dots i_\mu,i_*}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_{r\alpha+2}}\phi_{u+1}$$
$$+ \sum_{\nu \in N} a_\nu X\,div_{i_1} \dots X\,div_{i_\mu} C^{\nu,i_1\dots i_\mu}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+ \sum_{t \in \tilde{T}_1} a_t X\,div_{i_1} \dots X\,div_{i_{z_t}} C^{t,i_1\dots i_{z_t}}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1})$$
$$= \sum_{j \in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) = 0 \tag{4.72}$$

(notice the difference with (3.10) is that we are not including T_2, T_3, T_4 and T_1 has been replaced by $\tilde{T}_1$). Here the sublinear combination indexed in $\tilde{T}_1$ stands for a *generic* linear combination of the form $\sum_{t\in T_1}\dots$ described in the statement of Lemma 3.5. This is step 1.

In step 2 we use (4.72) to derive Proposition 2.1.

Special cases: There are two special cases which we will not consider here, but treat in [6]. The first special case is when $\sigma = 3$, $p = 3$,[64] and $n - 2\mu - 2u \leqslant 2$. The second special case is when $\sigma = 3$, $p = 2$, $\sigma + 2 = 1$ and $n = 2\mu + 2u$. For the rest of this section, we will be assuming that we do not fall under these special cases.

Proof of Step 1. Recall that $\sum_{t\in T_4}\dots$ appears only when the second critical factor is a simple factor of the from $\nabla^{(B)}\Omega_h$. In that case, we choose the factor $\nabla^{(B)}\Omega_x$ (referenced in the definition of the index set L^*_μ) to be the factor $\nabla^{(B)}\Omega_h$. (In other words we set $x = h$.) In order to show (4.72), we recall the hypothesis $L^*_\mu = \emptyset$ in Lemma 3.5. In other words, we are assuming that no tensor field $C^{l,i_1\dots i_\mu}_g$ in (2.3) has two free indices belonging to a factor $\nabla^{(2)}\Omega_h$.

Now, refer to the conclusion of Lemma 3.5 (in case A). In view of the above remark, it follows that none of the μ-tensor fields $\dot{C}^{l,i_2\dots i_\mu,i_*}_g\nabla_{i_2}\phi_{u+1}$ or $C^{\nu,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$ have an expression $\nabla_i\Omega_h\nabla^i\phi_{u+1}$. Thus, all tensor fields on the RHS of (3.10) with such an expression are indexed in T_4 (and have rank $z_t \geqslant \mu$, by definition).

Now, we will firstly focus on the sublinear combination

$$\sum_{t\in T_4} a_t X\,div_{i_1}\dots X\,div_{i_{z_t}} C^{t,i_1\dots i_{z_t}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

(recall $z_t \geqslant \mu$) *if it is non-zero*. (If it is zero we move onto the next stage.) We firstly seek to "get rid" of the tensor fields indexed in T_4. More precisely, we will show that we can write:

$$\begin{aligned}
&\sum_{t\in T_4} a_t X\,div_{i_1}\dots X\,div_{i_{z_t}} C^{t,i_1\dots i_{z_t}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\
&\quad=\sum_{t\in \tilde{T}_1} a_t X\,div_{i_1}\dots X\,div_{i_{z_t}} C^{t,i_1\dots i_{z_t}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\
&\qquad+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})
\end{aligned}\tag{4.73}$$

(with the same notational convention for the index set $\tilde{T}_1$ as above). If we can prove this, we will be reduced to showing step 1 under the assumption that $T_4 = \emptyset$.

Proof of (4.73). From (3.10) we straightforwardly derive that:

$$\sum_{t\in T_4} a_t X_*\,div_{i_1}\dots X_*\,div_{i_{z_t}} C^{t,i_1\dots i_{z_t}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0,\tag{4.74}$$

[64] Recall that p stands for the number of factors $\nabla^{(A)}\Omega_x$ in $\vec{\kappa}_{simp}$.

modulo complete contractions of length $\geqslant \sigma + u + 2$. Here $X_* div_i$ stands for the sublinear combination in $X div_i$ where ∇_i is additionally not allowed to hit the expression $\nabla_k \Omega_h \nabla^k \phi_{u+1}$.

We observe that (4.73) follows from Lemma 4.1 if $\sigma > 3$,[65] and from Lemma 4.2 if $\sigma = 3$.[66] □

Thus, we may now prove our claim under the additional assumption that $T_4 = \emptyset$. Next, we seek to "get rid" of the sublinear combination:

$$\sum_{t \in T_3} a_t X div_{i_1} \dots X div_{i_{z_t}} C_g^{t,i_1 \dots i_{z_t}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1})$$

in (3.10).

In particular, we will show that we can write:

$$\begin{aligned}
&\sum_{t \in T_3} a_t X div_{i_1} \dots X div_{i_{z_t}} C_g^{t,i_1 \dots i_{z_t}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) \\
&\quad = \sum_{t \in \tilde{T}_1} a_t X div_{i_1} \dots X div_{i_{z_t}} C_g^{t,i_1 \dots i_{z_t}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) \\
&\qquad + \sum_{j \in J} a_j C_g^j(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1})
\end{aligned} \tag{4.75}$$

($\sum_{t \in \tilde{T}_1} \dots$ stands for a generic linear combination as described in the conclusion of (4.72)).

Thus, if we can show the above, we may additionally assume that $T_3 = \emptyset$, in addition to our assumption that $T_4 = \emptyset$.

Proof of (4.75) when $\sigma > 3$. Break up T_3 into subsets $\{T_3^h\}_{h=1,\dots,p}$ according to the factor $\nabla \Omega_h$ that is differentiated only once. We will then show (4.75) for each of the index sets T_3^h separately.

To show this, we pick out the sublinear combination on (3.10) with a factor $\nabla \Omega_h$ (differentiated only once). This sublinear combination must vanish separately, hence we derive an equation:

$$\begin{aligned}
&\sum_{t \in T_3^h} a_t X_* div_{i_1} \dots X_* div_{i_{z_t}} C_g^{t,i_1 \dots i_{z_t}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) \\
&\qquad + \sum_{j \in J} a_j C_g^j(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) = 0,
\end{aligned} \tag{4.76}$$

modulo complete contractions of length $\geqslant \sigma + u + 2$; here as usual $X_* div_i$ stands for the sublinear combination in $X div_i$ where ∇_i is not allowed to hit the one factor $\nabla \Omega_h$.

Then, we see that (4.75) follows from (4.76) by applying Corollary 2 above (since $\mu \geqslant 4$ there are at least 2 derivative free indices for all maximal μ-tensor fields in our lemma assumption;

[65] By the definition of $\sum_{t \in T_4} \dots$ in the statement of Proposition 2.1, the assumptions of Lemma 4.1 are fulfilled.

[66] Notice that since we are assuming that (3.1) does not fall under the special case (described in the beginning of this subsection) the requirements of Lemma 4.2 are fulfilled.

therefore there exist at least two derivative free indices for each tensor field indexed in T_3, by weight considerations hence the requirements of Corollary 1 are fulfilled). $\square$

Proof of (4.75) when $\sigma = 3$. We apply the technique of the proof of Lemma 4.2 ("manually" constructing $X\,div$'s) to write out:

$$\sum_{t\in T_3} a_t X\,div_{i_1}\dots X\,div_{z_t} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$= (Const)_{1,*} X\,div_{i_1}\dots X\,div_{i_{A+1}} C_g^{*1,i_1\dots i_{A+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+ (Const)_{2,*} X\,div_{i_1}\dots X\,div_{i_A} C_g^{*2,i_1\dots i_A}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+ \sum_{t\in\tilde{T}_1} a_t X\,div_{i_1}\dots X\,div_{z_t} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+ \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}), \tag{4.77}$$

where the tensor fields $C_g^{*1,i_1\dots i_{A+1}}$, $C_g^{*2,i_1\dots i_A}$ are zero unless $p = 3$ or $\sigma_1 = 2$ or $\sigma_2 = 2$ or $\sigma_1 = \sigma_2 = 1$ (in the last case there will only be one tensor field $C_g^{*1,i_1\dots i_{A+1}}$ in the above). In those cases, they stand for the following tensor fields:

$$pcontr\big(\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t}\Omega_1 \otimes \nabla^{(B)}_{j_1\dots j_b y_1\dots y_r}\Omega_2 \otimes \nabla^{u_1}\phi_1 \otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla_{i_{A+1}}\Omega_3\big),$$

$$pcontr\big(\nabla^s\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t}\Omega_1 \otimes \nabla^{(B)}_{j_1\dots j_b y_1\dots y_r}\Omega_2 \otimes \nabla^{u_1}\phi_1 \otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla_s\Omega_3\big)$$

(here if $r \geqslant 2$ then $b = 0$; if $r \leqslant 1$ then $y = 2 - r$).

$$pcontr\big(\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t} R_{i_{a+1} j i_{a+2} l} \otimes \nabla^{(r)}_{y_1\dots y_r} R_{i_{a+3}}{}^{j}{}_{i_{a+4}}{}^{l} \otimes \nabla^{u_1}\phi_1 \otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla_{i_{A+1}}\Omega_1\big),$$

$$pcontr\big(\nabla^s\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t} R_{i_{a+1} j i_{a+2} l} \otimes \nabla^{(r)}_{y_1\dots y_r} R_{i_{a+3}}{}^{j}{}_{i_{a+4}}{}^{l} \otimes \nabla^{u_1}\phi_1 \otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla_s\Omega_1\big).$$

(In fact if $t + r = 0$ then there will be no $C_g^{*1,i_1\dots i_{A+1}}$ in (4.77).)

$$pcontr\big(S_*\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t} R_{i i_{a+1} i_{a+2} l} \otimes S_*\nabla^{(r)}_{y_1\dots y_r} R_{i' i_{a+3} i_{a+4}}{}^{l} \otimes \nabla^i\tilde{\phi}_1 \otimes \nabla^{i'}\tilde{\phi}_2 \otimes \nabla^{u_1}\phi_3$$
$$\otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla_{i_{A+1}}\Omega_1\big), \tag{4.78}$$

$$pcontr\big(\nabla^s S_*\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t} R_{i i_{a+1} i_{a+2} l} \otimes S_*\nabla^{(r)}_{y_1\dots y_r} R_{i' i_{a+3} i_{a+4}}{}^{l} \otimes \nabla^i\tilde{\phi}_1 \otimes \nabla^{i'}\tilde{\phi}_2 \otimes \nabla^{u_1}\phi_3$$
$$\otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla_s\Omega_1\big), \tag{4.79}$$

$$pcontr\big(S_*\nabla^{(X)}_{i_1\dots i_a u_1\dots u_t} R_{s i_{a+1} i_{a+2} l} \otimes \nabla^{(r)}_{y_1\dots y_r} R_{i' i_{a+3} i_{a+4}}{}^{l} \otimes \nabla^i\tilde{\phi}_1 \otimes \nabla^{i'}\tilde{\phi}_2 \otimes \nabla^{u_1}\phi_3$$
$$\otimes\dots\otimes \nabla^{y_r}\phi_{u+1} \otimes \nabla^s\Omega_1\big). \tag{4.80}$$

As in the proof of Lemma 4.2 we then derive that $(Const)_{1,*} = (Const)_{2,*} = 0$ in those cases; thus our claim follows in this case also. $\square$

Now, under the additional assumption that $T_3 = T_4 = \emptyset$, we focus on the sublinear combination

$$\sum_{t \in T_2} a_t X\, div_{i_1} \ldots X\, div_{i_{z_t}} C_g^{t,i_1 \ldots i_{z_t}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_{u+1})$$

in (3.10). We will show that we can write:

$$\begin{aligned}
&\sum_{t \in T_2} a_t X\, div_{i_1} \ldots X\, div_{i_{z_t}} C_g^{t,i_1 \ldots i_{z_t}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_{u+1}) \\
&\quad = \sum_{t \in \tilde{T}_1} a_t X\, div_{i_1} \ldots X\, div_{i_{z_t}} C_g^{t,i_1 \ldots i_{z_t}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_{u+1}) \\
&\qquad + \sum_{j \in J} a_j C_g^j(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_{u+1}),
\end{aligned} \tag{4.81}$$

where the notation is the same as in the statement of Lemma 3.5, and moreover $\sum_{t \in \tilde{T}_1} \ldots$ stands for a *generic* linear combination of the form described after (4.72). For each $t \in \tilde{T}_1$ we have $z_t \geqslant \mu$.

We will show a more general statement, for future reference.

Lemma 4.10. *Consider a linear combination of acceptable tensor fields,*

$$\sum_{l \in L_1} a_l C_g^{l,i_1 \ldots i_{z_l}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \phi_{u+1}$$

with a u-simple character $\vec{\kappa}_{simp}$ ($\sigma \geqslant 3$) *and with a* $(u+1)$*-simple character* $\vec{\kappa}^+_{simp}$, *where in addition we are assuming that if* $\nabla_{i_1}\phi_{u+1}$ *is contracting against a factor* $\nabla^{(m)} R_{ijkl}$ *then it is contracting against a derivative index, whereas if it is contracting against a factor* $S_* \nabla^{(\nu)} R_{ijkl}$ *it must be contracting against one of the indices* $_{r_1}, \ldots, _{r_\nu}, _j$. *(This is the defining property of the* $(u+1)$*-simple character* $\vec{\kappa}^+_{simp}$.*)*

Consider another linear combination of acceptable tensor fields,

$$\sum_{l \in L_2} a_l C_g^{l,i_1 \ldots i_{z_l}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} \phi_{u+1}$$

with a u-simple character $\vec{\kappa}_{simp}$ *and weak* $(u+1)$*-character equal to* $Weak(\vec{\kappa}^+_{simp})$, *where in addition* $\nabla_{i_1}\phi_{u+1}$ *is either contracting against an internal index in some factor* $\nabla^{(m)} R_{ijkl}$ *or an index* $_k$ *or* $_l$ *in a factor* $S_* \nabla^{(\nu)} R_{ijkl}$. *We moreover assume that each* $l \in L_2$ *we have* $z_l \geqslant \gamma$, *for some number* γ, *and denote by* $L_2^\gamma \subset L_2$ *the index set of the tensor fields with order* γ.

Assume that:

$$X\,div_{i_2}\ldots X\,div_{i_{z_l}}\sum_{l\in L_1}a_lC_g^{l,i_1\ldots i_{z_l}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+X\,div_{i_2}\ldots X\,div_{i_{z_l}}\sum_{l\in L_2}a_lC_g^{l,i_1\ldots i_{z_l}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$=\sum_{j\in J}a_jC_g^{j,i_1}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}, \tag{4.82}$$

where each C_g^{j,i_1} is u-subsequent to $\vec{\kappa}_{simp}$. Furthermore assume that the above equation falls under the inductive assumption of Proposition 2.1 *(with regard to the parameters weight, σ, Φ, p). Furthermore, we additionally assume that none of the tensor fields $C_g^{l,i_1\ldots i_{z_l}}$ of minimum rank in* (4.82)[67] *are "forbidden" in the sense of Proposition* 2.1.

Our first claim is then that there exists a linear combination of $(\gamma+1)$-tensor fields, $\sum_{l\in L_2'}a_lC_g^{l,i_1\ldots i_{\gamma+1}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}$ with u-simple character $\vec{\kappa}$ and weak $(u+1)$-character equal to Weak$(\vec{\kappa}^+_{simp})$, where in addition $\nabla_{i_1}\phi_{u+1}$ is either contracting against an internal index in some factor $\nabla^{(m)}R_{ijkl}$ or an index $_k$ or $_l$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$, so that:

$$\sum_{l\in L_2^\gamma}a_lC_g^{l,i_1\ldots i_\gamma}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\ldots\nabla_{i_\gamma}\upsilon$$
$$-X\,div_{i_\gamma+1}\sum_{l\in L_2'}a_lC_g^{l,i_1\ldots i_{\gamma+1}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\ldots\nabla_{i_\gamma}\upsilon$$
$$+\sum_{l\in\overline{L}_1}a_lC_g^{l,i_1\ldots i_\gamma}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\ldots\nabla_{i_\gamma}\upsilon$$
$$+\sum_{j\in J}a_jC_g^{j,i_1\ldots i_\gamma}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\ldots\nabla_{i_\gamma}\upsilon; \tag{4.83}$$

here each $C_g^{j,i_1\ldots i_\gamma}$ is u-subsequent to $\vec{\kappa}_{simp}$. The tensor fields indexed in $\overline{L}_1$ are like the ones indexed in L_1 in (4.82)*, but in addition each $z_l\geqslant\gamma$.*

Our second claim is that assuming (4.82) *we can write:*

$$X\,div_{i_2}\ldots X\,div_{i_{z_l}}\sum_{l\in L_2}a_lC_g^{l,i_1\ldots i_{z_l}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$=\sum_{l\in\overline{L}_1}a_lX\,div_{i_2}\ldots X\,div_{i_{z_l}}C_g^{l,i_1\ldots i_{z_l}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}$$
$$+\sum_{j\in J}a_jC_g^{j,i_1}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\phi_{u+1}, \tag{4.84}$$

where C_g^{j,i_1} is u-subsequent to $\vec{\kappa}^+_{simp}$.

[67] I.e. of rank γ.

Our third claim is that if γ is the minimum rank among all tensor fields in $L_1 \cup L_2$ in our assumption and L_1^γ, L_2^γ their respective index sets, then there exists a linear combination of $(\gamma+1)$-tensor fields, $\sum_{l\in L_3} a_l C_g^{l,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ with u-simple character $\vec{\kappa}$ and weak $(u+1)$-character equal to $Weak(\vec{\kappa}^+_{simp})$ so that:

$$\sum_{l\in L_1^\gamma\cup L_2^\gamma} a_l C_g^{l,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon$$
$$- X\,div_{i_{\gamma+1}}\sum_{l\in L^3} a_l C_g^{l,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon$$
$$=\sum_{j\in J} a_j C_g^{j,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon; \tag{4.85}$$

here each $C_g^{j,i_1\dots i_\gamma}$ is u-subsequent to $\vec{\kappa}_{simp}$.

We observe that if we can show the above, then our claim (4.81) follows from the second step of this lemma.

Proof of Lemma 4.10. We firstly remark that in proving Lemma 4.10 we will use Lemma A.2, which is stated and proven in Appendix A of this paper. We also easily observe that the third claim above follows from the first two. So we now prove the first two claims in that lemma:

Proof of the second claim of Lemma 4.10. We now show that the second claim follows from the first one.

We will show this by induction. We assume that $\min_{l\in L_2} z_t = \gamma' \geqslant \gamma$. We denote the index set of those tensor fields by $L_2^{\gamma'} \subset L_2$. Then, using the first claim[68] and making the $\nabla\upsilon$s into $X\,div$s, we derive that we can write:

$$\sum_{l\in L_2^{\gamma'}} a_l X\,div_{i_1}\dots X\,div_{i_\gamma} C_g^{t,i_1,\dots,i_{z_l}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$=\sum_{l\in L_2'} a_l X\,div_{i_1}\dots X\,div_{i_{z_t}} C_g^{l,i_1\dots i_{z_l}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+\sum_{l\in L_1} a_l X\,div_{i_1}\dots X\,div_{i_{z_l}} C_g^{l,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}), \tag{4.86}$$

where the tensor fields indexed in L_2' are of the exact same form as the ones indexed in L_2 in (3.5), with the additional property that $z_l \geqslant \gamma'+1$. We notice that since we are dealing with

[68] Provided that there are no terms indexed in $L_2^{\gamma'}$ which are forbidden.

tensor fields of a given weight $-n$, iteratively repeating this step we derive our second step. (*Note:* If at the last step we encounter a "forbidden case" then clearly $\gamma' > \gamma$—we then apply Lemma A.2 below with $\Phi = 1$.)

Proof of the first claim of Lemma 4.10. The proof requires only our inductive assumption on Corollary 1. We have two cases to consider: Firstly, when the factor $\nabla\phi_{u+1}$ is contracting (in $\vec{\kappa}^+_{simp}$) against an internal index (say ${}_i$ with no loss of generality) of a factor $\nabla^{(m)}R_{ijkl}$. Secondly, when the factor $\nabla\phi_{u+1}$ is contracting against an index ${}_k$ of a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

Proof of first claim of Lemma 4.10 in the first case. In the first case, we define an operation *CutSym* that acts on the tensor fields indexed in L_2 by replacing the expression $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^{r_1}\phi_{t_1}\dots\nabla^{r_a}\phi_{t_a}\nabla^i\phi_{u+1}$ by an expression $S_*\nabla^{(m-a)}_{r_{a+1}\dots r_n}R_{ijkl}\nabla^i\phi_{u+1}$. We observe that the tensor fields that arise via this operation have a given simple character which we will denote by $\vec{\kappa}_{cut}$. For each $l \in L_2$ we denote by

$$C^{l,i_1\dots i_{z_l}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$

the tensor field that we obtain by applying this operation.

We also define the operation *CutSym* to act on the tensor fields indexed in L_1 by replacing the expression $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^{r_1}\phi_{t_1}\dots\nabla^{r_a}\phi_{t_a}\nabla^{r_b}\phi_{u+1}$ by a factor $\nabla^{(m-1)}_{r_{a+1}\dots r_m}R_{ijkl}\nabla^{r_b}\phi_{u+1}$. Now, by applying the eraser to the factors $\nabla^{r_1}\phi_{t_1}\dots\nabla^{r_a}\phi_{t_a}$ and S_*-symmetrizing, we may apply *CutSym* to (4.82) and derive an equation:

$$\begin{aligned} &X\,div_{i_2}\dots X\,div_{i_{z_l}}\sum_{l\in L_2}a_l C^{l,i_1\dots i_{z_l}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1} \\ &\quad=\sum_{j\in J}a_j C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}, \end{aligned} \tag{4.87}$$

where each C^j_g is $(u-a)$-subsequent to $\vec{\kappa}_{cut}$. We may then apply Corollary 1 to the above.[69] This follows since either the weight in (4.87) is $-n', n' < n$ (this occurs when we erase factors $\nabla\phi_t$ upon performing the operation *CutSym*), or the weight is $-n$ and there are $u+1$ factors $\nabla\phi_h$ in (4.87). Thus, our inductive assumption of Corollary 1 applies to (4.87).

Thus, by direct application of Corollary 1 (which we are now inductively assuming because either the weight is $> -n$ or there are $(u+1)$ factors $\nabla\phi$) to (4.87) we derive that there is an acceptable linear combination of $(\gamma+1)$-tensor fields with a simple character $\vec{\kappa}_{cut}$, say

$$\sum_{x\in X}a_x C^{x,i_1\dots i_{\gamma+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1},$$

so that:

[69] Corollary 1 may be applied by virtue of our assumptions on various terms in our lemma assumption not being "forbidden". This ensures that the terms of minimum rank in (4.87) are not "forbidden" in the sense of Corollary 1.

$$\sum_{l\in L_2^{\gamma}} a_l C_g^{l,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon$$
$$-\sum_{x\in X} a_x C_g^{x,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon$$
$$=\sum_{j\in J} a_j C_g^{j,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_{r_{a_1}},\dots,\hat{\phi}_{r_{a_t}},\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon, \quad (4.88)$$

where each tensor field C_g^j is subsequent to $\vec{\kappa}_{cut}$.

Now, we define an operation *Add* that acts on the tensor fields above by replacing the expression $S_*\nabla^{(m-a)}_{r_{a+1}\dots r_n}R_{ijkl}\nabla^i\phi_{u+1}$ by an expression

$$\nabla_{r_1\dots r_a}S_*\nabla^{(m-a)}_{r_{a+1}\dots r_m}R_{ijkl}\nabla^{r_1}\phi_{t_1}\dots\nabla^{r_a}\phi_{t_a}\nabla^i\phi_{u+1}.$$

In case $\nabla\phi_{u+1}$ is contracting against some derivative index in some $\nabla^{(m)}R_{ijkl}$, it adds on the factor $\nabla^{(m)}R_{ijkl}$ against which $\nabla\phi_{u+1}$ is contracting a derivative indices and contracts them against factors $\nabla\phi_{a_1},\dots,\phi_{a_t}$. By applying the operation *Add* to (4.88) we derive our desired equation (4.83). □

The second case is treated in a similar fashion. We now define a formal operation *CutY* as follows: *CutY* acts on the tensor fields indexed in L_2 by replacing the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ir_{\nu+1}kl}\nabla^{r_1}\phi'_{t_1}\dots\nabla^{r_a}\phi'_{t_a}\nabla^i\tilde{\phi}_{t_{a+1}}\nabla^k\phi_{u+1}$ by an expression $\nabla^{(\nu-a+2)}_{r_a\dots r_{n+1}l}Y\nabla^{r_a}\phi'_{t_a}$ (if there is at least one factor $\nabla\phi'$ contracting against our factor $S_*\nabla^{(\nu)}R_{ijkl}$; if there is no such factor we replace $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ir_{\nu+1}kl}\nabla^i\tilde{\phi}_{t_{a+1}}\nabla^k\phi_{u+1}$ by $\nabla^{(\nu+2)}_{r_1\dots r_{\nu+1}l}Y$. We will denote the tensor field thus obtained by $C_g^{l,i_1\dots i_{z_l}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_{a+1}},\dots,\phi_u)$. (Observe that it is acceptable if we treat the function Y as a function Ω_{p+1}. We observe that all the tensor fields that arise thus have a given simple character which we will denote by $\vec{\kappa}_{cut}$.) We also define *CutY* to act on the tensor fields indexed in L_1 by replacing them by zero. Finally, it follows easily that the operation *CutY* either annihilates a given complete contraction C_g^j, or replaces it by a complete contraction that is subsequent to $\vec{\kappa}_{cut}$.

Now, by virtue of Lemma 4.4 and the "Eraser" (defined in the Appendix of [2]), we see that applying *CutY* to (4.82) produces a true equation which can be written as:

$$X\,div_{i_2}\dots X\,div_{i_{z_l}}\sum_{l\in L_2} a_l C_g^{l,i_1\dots i_{z_l}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_{a+1}},\dots,\phi_u)$$
$$=\sum_{k\in K} a_k C_g^{k,i_1}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_{a+1}},\dots,\phi_u), \quad (4.89)$$

where each C_g^k is simply subsequent to $\vec{\kappa}_{cut}$. Thus, by direct application of Corollary 1 to (4.87),[70] we derive that there is an acceptable linear combination of $(\gamma+1)$-tensor fields with a simple character $\vec{\kappa}_{cut}$, $\sum_{x\in X} a_x C_g^{x,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_p,Y\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_a},\dots,\phi_u)$, so that:

[70] The observation in the previous footnote still applies—by virtue of the assumptions imposed in our lemma hypothesis, (4.87) does not fall under a "forbidden case".

$$\sum_{l\in L_2^\gamma} a_l C_g^{l,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_a},\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon$$
$$-\sum_{x\in X} a_x X\,div_{i_{\gamma+1}} C_g^{x,i_1\dots i_{\gamma+1}}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_a},\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon$$
$$=\sum_{k\in K} a_k C_g^{k,i_1\dots i_\gamma}(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{t_1},\dots,\hat{\phi}_{t_a},\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\gamma}\upsilon, \tag{4.90}$$

where each tensor field C_g^k is subsequent to $\vec{\kappa}_{cut}$. (Note that the LHS in (4.89) has weight $> -n$, hence Corollary 1 applies, thanks to our inductive assumption.)

Now, we define a formal operation UnY that acts on the tensor fields above by replacing the expression $\nabla^{(B)}_{t_1\dots t_B}Y$, $B\geqslant 2$ by an expression $\nabla^{(B-2+a)}_{r_1\dots r_a t_1\dots t_{B-2+a}} R_{ijkl}\nabla^{r_1}\phi_{t_1}\dots\nabla^{r_a}\phi_{t_a}\nabla^i\phi_{t_{a+1}}\cdot\nabla^k\phi_{u+1}$. By applying the operation UnY to (4.90) (and repeating the permutations by which (4.90) is made formally zero, modulo introducing correction terms by virtue of the Bianchi identities—see (4.19), (4.20), (4.21)) we derive our desired equation (4.83). □

This completes the proof of step 1 (in the derivation of Proposition 2.1 (in case III) from Lemma 3.5. □

Proof of Step 2 (in the derivation of Proposition 2.1 (in case III)) from Lemma 3.5. We consider (4.72) (where all the tensor fields are now acceptable, by definition). Recall that the $(u+1,\mu-1)$-refined double characters that correspond to the index sets L^z, $z\in Z'_{Max}$ in (4.72) (we have denoted them by $\vec{L}^{z,\sharp}$) are the maximal ones. Now, we can apply our inductive assumption of Proposition 2.1 to (4.72)[71]:

We derive that for each $z\in Z'_{Max}$, there is a linear combination of acceptable μ-tensor fields (which satisfy the extra restriction if it is applicable),

$$\sum_{p\in P'} a_p C_g^{p,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$$

with a $(u+1,\mu-1)$-refined double character $\vec{L}^{z,\sharp}$, so that for any $z\in Z'_{Max}$:

$$\binom{\alpha}{2}\sum_{l\in L^z} a_l\sum_{r=1}^{k-1}\dot{C}_g^{l,i_1\dots\widehat{i_{r\alpha+1}}\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_*}\upsilon$$
$$+\sum_{p\in P'} a_p X\,div_{i_\mu} C_g^{p,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon$$
$$=\sum_{k\in K} a_k C_g^k\big(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\upsilon^{\mu-1}\big), \tag{4.91}$$

[71] The inductive assumption of Proposition 2.1 applies here since we have weight $-n$ but have an extra factor $\nabla\phi_{u+1}$. Observe that the $(\mu-1)$-tensor fields in that equation have no special free indices, hence there is no danger of "forbidden cases".

modulo complete contractions of length $\geqslant \sigma + u + \mu + 1$. Here each C^k is (simply or doubly) subsequent to each $\vec{L}^{z,\sharp}$, $z \in Z'_{Max}$.

We then define a formal operation Op that acts on the tensor fields in the above by performing two actions: Firstly, we pick out a derivative index in the critical factor (the *unique* factor that is contracting against the most factors $\nabla\upsilon$) that contracts against a factor $\nabla\upsilon$ and erase it. Secondly, we then add a derivative index ∇_{i_+} onto the A-crucial factor and contract it against the above factor $\nabla\upsilon$.

Let us observe that this operation is well-defined, and then see that it produces a true equation: The only thing that could make this operation not well-defined is if no factor $\nabla\upsilon$ is contracting against a derivative index in the critical factor (this can only be the case for factors $S_*\nabla^{(\nu)}R_{ijkl}$ with $\nu = 0$). But that cannot happen: Recall the critical factor must start out with at least two free indices (none of them special), and then we add another derivative index onto it. Thus in all complete contractions in (4.91) there are at least three factors $\nabla\upsilon$ contracting against (non-special) indices in the critical factor. Thus, our operation Op is well-defined. By the same reasoning, observe that our operation Op produces acceptable tensor fields.

We then set $\phi_{u+1} = \upsilon$. We have observed that Op is well defined, and we see that after we set $\phi_{u+1} = \upsilon$, we will have that for each $l \in L^z$, $z \in Z'_{Max}$:

$$\begin{aligned} &Op\big[\dot{C}_g^{l,i_1\dots\widehat{i_{r\alpha+1}}\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_*}\upsilon\big] \\ &\quad= C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon. \end{aligned} \tag{4.92}$$

Hence, applying Op to (4.91) (which produces a true equation since (4.91) holds formally) gives us step 2 and thus we derive the claim of Proposition 2.1 in case III from Lemma 3.5. □

Appendix A

A.1. A weak substitute for Proposition 2.1 in the "forbidden cases"

We present a "substitute" of sorts of Proposition 2.1 in the "forbidden cases". This "substitute" (Lemma A.2 below) will rely on a generalized version of the Lemma 4.10, Lemma A.1 which is stated below but proven in [6]. The generalized version asserts that the claim of Lemma 4.10 remains true, for the general case where rather than one "additional" factor $\nabla\phi_{u+1}$ we have $\beta \geqslant 3$ "additional" factors $\nabla\phi_{u+1},\dots,\nabla\phi_{u+\beta}$. Moreover, in that case there are no "forbidden cases".

Lemma A.1. *Let*

$$\sum_{l\in L_1} a_l C_g^{l,i_1\dots i_\mu,i_{\mu+1}\dots i_{\mu+\beta}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

$$\sum_{l\in L_2} a_l C_g^{l,i_1\dots i_{b_l},i_{b_l+1}\dots i_{b_l+\beta}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

stand for two linear combinations of acceptable tensor fields in the form (2.2)*, with a u-simple character $\vec{\kappa}_{simp}$. We assume that the terms indexed in L_1 have rank $\mu+\beta$, while the ones indexed in L_2 have rank greater than $\mu+\beta$.*

Assume an equation:

$$\sum_{l\in L_1} a_l X\,div_{i_1}\dots X\,div_{i_\mu} C_g^{l,i_1\dots i_{\mu+\beta}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\dots\nabla_{i_\beta}\phi_{u+\beta}$$
$$+\sum_{l\in L_2} a_l X\,div_{i_1}\dots X\,div_{i_{b_l}} C_g^{l,i_1\dots i_{b_l+\beta}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\dots\nabla_{i_\beta}\phi_{u+\beta}$$
$$+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+\beta})=0, \tag{A.1}$$

modulo terms of length $\geqslant \sigma+u+\beta+1$. *Furthermore, we assume that the above equation falls under the inductive assumption of Proposition* 2.1 *in* [5] *(with regard to the parameters weight,* σ, Φ, p*). We are not excluding any "forbidden cases".*

We claim that there exists a linear combination of $(\mu+\beta+1)$*-tensor fields in the form* (2.2) *with u-simple character* $\vec{\kappa}_{simp}$ *and length* $\sigma+u$ *(indexed in* H *below) such that:*

$$\sum_{l\in L_1} a_l C_g^{l,i_1\dots i_{\mu+\beta}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\dots\nabla_{i_\beta}\phi_{u+\beta}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$+\sum_{h\in H} a_h X\,div_{i_{\mu+\beta+1}} C_g^{l,i_1\dots i_{\mu+\beta+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\dots\nabla_{i_\beta}\phi_{u+\beta}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$+\sum_{j\in J} a_j C_g^j\big(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+\beta},\upsilon^\mu\big)=0, \tag{A.2}$$

modulo terms of length $\geqslant \sigma+u+\beta+\mu+1$. *The terms indexed in* J *here are u-simply subsequent to* $\vec{\kappa}_{simp}$.

A note and a notational convention before we state our next lemma: We observe that if some of the μ-tensor fields of maximal refined double character in (2.3) are "forbidden", then necessarily *all* tensor fields in (2.3) must have rank μ (in other words $L_{>\mu}=\emptyset$). This follows from weight considerations. *Also* all the μ-tensor fields must have each of the μ free indices belonging to a different factor. This follows from the definition of maximal refined double character.

We introduce the notational convention needed for our lemma. For each tensor field $C_g^{l,i_1\dots i_\mu}$ appearing in (2.3) we will consider its product with an auxiliary function Φ, $C_g^{l,i_1\dots i_\mu}\cdot\Phi$. In that context $X\,div_{i_a}[C_g^{l,i_1\dots i_\mu}\cdot\Phi]$ will stand for the sublinear combination in where ∇^{i_a} is not allowed to hit the factor to which $_{i_a}$ belongs (but it *is* allowed to hit the function Φ).

Lemma A.2. *Assume Eq.* (2.3)*, under the additional assumption that some of the tensor fields of maximal refined double character in* L_μ *are "forbidden", in the sense of Definition* 2.12. *Denote by* $\vec{\kappa}_{simp}$ *the u-simple character of the tensor field in* (2.3).

We then claim that there is then a linear combination of acceptable μ*-tensor fields with a u-simple character* $\vec{\kappa}_{simp}$ *indexed in* H *below so that:*

$$\sum_{l\in L_\mu} a_l X\,div_{i_1}\dots X\,div_{i_\mu}\big[C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\cdot\Phi\big]$$
$$=\sum_{h\in H} a_h X\,div_{i_2}\dots X\,div_{i_\mu}\big[C_g^{h,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\Phi\big]$$

$$+\sum_{j\in J} a_j X\,div_{i_1}\ldots X\,div_{i_{\mu-1}}\big[C^{j,i_1\ldots i_{\mu-1}}_g(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\cdot\Phi\big] \tag{A.3}$$

(modulo longer terms); here the terms indexed in J are acceptable $(\mu-1)$-tensor fields in the form (2.1) which are simply subsequent to $\vec{\kappa}_{simp}$.

Proof. Pick out the factor $T_1=S_*R_{ijkl}\nabla^i\tilde{\phi}_1$. Let $L^A_\mu\subset L_\mu$ stand for the index set of terms in which contain a free index in this special factor and let $L^B_\mu\subset L_\mu$ stand for the index set of terms with no free index in that factor. We assume wlog that for each $l\in L^A_\mu$ the free index that belongs to the factor T_1 is ${}_{i_1}$.

We will prove the following statements:

$$\sum_{l\in L^A_\mu} a_l X\,div_{i_1} C^{l,i_1\ldots i_\mu}_g\nabla_{i_2}\upsilon\ldots\nabla_{i_\mu}\upsilon$$
$$=\sum_{l\in L'^B_\mu} a_l X\,div_{i_1} C^{l,i_1\ldots i_\mu}_g\nabla_{i_2}\upsilon\ldots\nabla_{i_\mu}\upsilon$$
$$+\sum_{t\in T} a_t C^{t,i_2\ldots i_\mu}_g\nabla_{i_2}\upsilon\ldots\nabla_{i_\mu}\upsilon+\sum_{j\in J} a_j C^{j,i_2\ldots i_\mu}_g\nabla_{i_2}\upsilon\ldots\nabla_{i_\mu}\upsilon, \tag{A.4}$$

where the tensor fields indexed in L'^B_μ are just like the tensor fields indexed in L^B_μ, but the free index ${}_{i_1}$ does not belong to the factor T_1. The tensor fields indexed in T are acceptable tensor fields of rank $\mu-1$, with a simple character $\vec{\kappa}_{simp}$, and moreover they have a factor $S_*\nabla R_{ijkl}$ (with one derivative) which does not contain a free index.

We will prove (A.4) momentarily. Let us now check how it implies our claim: We convert the factors $\nabla\upsilon$'s into $X\,div$'s (we are using the last lemma in the Appendix of [2] here), and replace into our lemma hypothesis, to derive a new equation:

$$\sum_{t\in T} a_t X\,div_{i_2}\ldots X\,div_{i_\mu} C^{t,i_2\ldots i_\mu}_g+\sum_{l\in L^B_\mu\cup L'^B_\mu} X\,div_{i_1}\ldots X\,div_{i_\mu} C^{l,i_1\ldots i_\mu}_g+\sum_{j\in J} a_j C^j_g. \tag{A.5}$$

We next pick out the sublinear combination of terms in (A.5) with a factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ (no derivatives); this sublinear combination clearly vanishes separately, so we derive:

$$\sum_{l\in L^B_\mu\cup L'^B_\mu} X_*\,div_{i_1}\ldots X_*\,div_{i_\mu} C^{l,i_1\ldots i_\mu}_g+\sum_{j\in J} a_j C^j_g=0. \tag{A.6}$$

(Here $X_*\,div_i[\ldots]$ stands for the sublinear combination in $X\,div_i[\ldots]$ where ∇^i is not allowed to hit the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$. We then define a formal operation $Op[\ldots]$ which acts on the terms above by replacing the expression $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ by an expression $\nabla_j\omega\nabla_k\omega\nabla_l\upsilon-\nabla_j\omega\nabla_l\omega\nabla_k\upsilon$; denote the resulting $(u-1)$-simple character (which keeps track of the indices $\nabla\tilde{\phi}_2,\ldots,\nabla\tilde{\phi}_u$) by $\vec{\kappa}'_{simp}$.) Observe that this produces a new true equation:

$$\sum_{l\in L^B_\mu\cup L'^B_\mu} X\,div_{i_1}\ldots X\,div_{i_\mu} Op[C]^{l,i_1\ldots i_\mu}_g+\sum_{j\in J} a_j C^j_g=0, \tag{A.7}$$

where the terms C_g^j are simply subsequent to $\vec{\kappa}'_{simp}$. We can then apply Lemma A.1 to the above (the above falls under the inductive assumption of (the generalized version of) Lemma 4.10 because the terms above have $\sigma_1+\sigma_2+p=\sigma-1$). We derive that:

$$\sum_{l\in L^B_\mu\cup L'^B_\mu} Op[C]^{l,i_1\dots i_\mu}_g \nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=0. \tag{A.8}$$

Now, we formally replace each expression $\nabla_a\omega\nabla_b\omega$, $\nabla_c\upsilon$ by an expression $S_*R_{i(ab)c}\nabla^i\tilde{\phi}_1$ and derive that:

$$\sum_{l\in L^B_\mu\cup L'^B_\mu} C^{l,i_1\dots i_\mu}_g \nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=0. \tag{A.9}$$

Thus, we may return to (A.5) and erase the sublinear combination in $L^B_\mu\cup L'^B_\mu$. We then pick out the sublinear combination in that equation with a factor $S_*\nabla_aR_{ijkl}\nabla^i\tilde{\phi}_1$. We then derive a new true equation:

$$\sum_{t\in T} a_t X_*div_{i_2}\dots X_*div_{i_\mu}C^{t,i_2\dots i_\mu}_g+\sum_{j\in J}a_jC^j_g \tag{A.10}$$

($X_*div_i\dots$ now means that ∇^i is not allowed to hit the factor $S_*\nabla_aR_{ijkl}$). We then define a formal operation $Op'[\dots]$ which acts on the terms above by replacing the expression $S_*\nabla_aR_{ijkl}\nabla^{i_1}\tilde{\phi}_1$ by an expression $\nabla_a\omega\nabla_j\omega\nabla_k\omega\nabla_l\upsilon-\nabla_a\omega\nabla_j\omega\nabla_l\omega\nabla_k\upsilon$; denote the resulting $(u-1)$-simple character (which keeps track of the indices $\nabla\tilde{\phi}_2,\dots,\nabla\tilde{\phi}_u$) by $\vec{\kappa}'_{simp}$. Then by the same argument as above, we derive that:

$$\sum_{t\in T} a_t Op'[C]^{t,i_2\dots i_\mu}_g \nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon=0, \tag{A.11}$$

and therefore:

$$\sum_{t\in T} a_t C^{t,i_2\dots i_\mu}_g \nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon=0. \tag{A.12}$$

Thus, we derive our claim by just multiplying Eqs. (A.4), (A.9), (A.12) by Φ, converting the $\nabla\upsilon$'s into $X\,div$'s (we are here applying the relevant lemma from the Appendix of [2]),[72] and then adding the resulting equations.

Thus, matters are reduced to proving (A.4). We do this as follows: Refer to our lemma assumption and pick out the sublinear combination of terms with a factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ (with no derivatives). This sublinear combination vanishes separately, thus we derive a new true equation:

[72] Recall that in this setting the derivative ∇^i in each $X\,div_i$ *is* allowed to hit the factor Φ.

$$\sum_{l\in L_1^A} a_l X\, div_{i_1} X_* div_{i_2} \ldots X_* div_{i_\mu} C_g^{l,i_1\ldots i_\mu} + \sum_{l\in L_1^B} a_l X_* div_{i_1} \ldots X_* div_{i_\mu} C_g^{l,i_1\ldots i_\mu} + \sum_{j\in J} a_j C_g^j = 0. \tag{A.13}$$

Again, applying the operation Op (defined above) to (A.13) we derive a new true equation:

$$\sum_{l\in L_1^A} a_l X_* div_{i_2} \ldots X_* div_{i_\mu} \{X\, div_{i_1} Op[C]_g^{l,i_1\ldots i_\mu}\} + \sum_{l\in L_1^B} a_l X_* div_{i_1} \ldots X_* div_{i_\mu} Op[C]_g^{l,i_1\ldots i_\mu} + \sum_{j\in J} a_j C_g^j = 0. \tag{A.14}$$

Here $X\, div_{i_1} Op[C]_g^{l,i_1\ldots i_\mu}$ stands for the sublinear combination where the derivative ∇^i is not allowed to hit any of the factors $\nabla\phi_h$ nor any of the factors $\nabla\omega, \nabla\upsilon$. In fact, if we treat the $X\, div_i Op[C]_g^{l,i_1\ldots i_\mu}$ as a sum of $(\mu-1)$-tensor fields (so we forget its $X\, div$-structure). We then apply the inductive assumption of Lemma 4.10 to derive that there exists a linear combination of μ-tensor fields with a $(u-1)$-simple character $\vec{\kappa}'_{simp}$, such that:

$$\sum_{l\in L_1^A} a_l \{X\, div_{i_1} Op[C]_g^{l,i_1\ldots i_\mu}\} \nabla_{i_2}\upsilon \ldots \nabla\upsilon_{i_\mu} + \sum_{h\in H} a_l X_* div_{i_\mu} C_g^{h,i_1\ldots i_\mu} \nabla_{i_2}\upsilon \ldots \nabla\upsilon_{i_\mu} + \sum_{j\in J} a_j C_g^j = 0. \tag{A.15}$$

Now, we act on the above by another formal operation $Op^{-1}[\ldots]$ which replaces each expression $\nabla_a\omega\nabla_b\omega\nabla_c\upsilon$ by $S_* R_{i(ab)c}\nabla^i\tilde{\phi}_1$. The result precisely is our desired (A.4). $\Box$

A.2. A postponed claim

We let M stand for the number of free indices in the critical factor, for the terms of maximal refined double character in (3.1). We denote by $L_{\mu,*} \subset \bigcup_{z\in Z'_{Max}} L^z$ the index set of μ-tensor fields in (3.1) with M free indices in the critical factor and with only one index (the index $_l$) in the critical factor contracting against another factor, *in particular against a special index in some (simple) factor* $S_*\nabla^{(\rho)}R_{abcd}$.[73] We will show that:

$$\sum_{l\in L_{\mu,*}} a_l C_g^{l,i_1\ldots i_\mu}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\upsilon\ldots\nabla_{i_\mu}\upsilon - \sum_{h\in H} a_h X\, div_{i_{\mu+1}} C_g^{h,i_1\ldots i_{\mu+1}}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\upsilon\ldots\nabla_{i_\mu}\upsilon = \sum_{t\in T} a_t C_g^{t,i_1\ldots i_\mu}(\Omega_1,\ldots,\Omega_p,\phi_1,\ldots,\phi_u)\nabla_{i_1}\upsilon\ldots\nabla_{i_\mu}\upsilon, \tag{A.16}$$

[73] Denote this other factor by T' (while the critical factor will be denoted by T_*).

where the tensor fields in the RHS have all the features of the tensor fields in the first line but in addition the index $_l$ in the critical factor is contracting against a non-special index. If we can show (A.16), it then follows that the "delicate assumption" can be made with no loss of generality.

Proof of (A.16). We divide the index set $L_{\mu,*}$ according to *which* factor $S_*\nabla^{(\rho)}R_{abcd}$ the index $_l$ in the critical factor is contracting against: We say that $l\in L_{\mu,*,k}, k\in K$ if and only if the index $_l$ is contracting against a special index in the factor $S_*\nabla^{(\rho)}R_{ijcd}\nabla^i\tilde{\phi}_k$ (denote this factor by T'_k)—say the index $_l$ in T'_k.

We prove (A.16) for the terms in $L_{\mu,*,k}$. Clearly, if we can prove this then the whole of (A.16) will follow. We denote by $\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_1,\dots,\phi_u)$ the tensor field that arises from $C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)$ by replacing the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}\times S_*\nabla^{(\rho)}_{y_1\dots y_\rho}R_{i'j'k'}{}^l\nabla^i\tilde{\phi}_1\nabla^{i'}\tilde{\phi}_k$ by $\nabla^{(\nu+2)}_{r_1\dots r_\nu jk}Y_1\nabla^{(\rho+2)}_{y_1\dots y_\rho j'k'}Y_2$; denote the resulting simple character by $Cut[\vec{\kappa}_{simp}]$. Considering the second conformal variation of (3.1) and pick out the terms of length $\sigma+u$ with the factors $\nabla\tilde{\phi}_1,\nabla\tilde{\phi}_k$ contracting against each other, we derive a new true equation:

$$\Bigg[\sum_{l\in L_{\mu,*,k}}a_l X\,div_{i_1}\dots X\,div_{i_\mu}\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)$$
$$+\sum_{h\in H}a_h X\,div_{i_1}\dots X\,div_{i_a}C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)$$
$$+\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)\Bigg]\nabla^s\tilde{\phi}_1\nabla_s\tilde{\phi}_k=0. \tag{A.17}$$

The terms indexed in H have length $>\mu$ and are acceptable with a simple character $Cut[\vec{\kappa}_{simp}]$. The complete contractions indexed in J are simply subsequent to $Cut[\vec{\kappa}_{simp}]$.

Now, we apply our inductive assumption of Corollary 1 to the above,[74] and we pick out the sublinear combination of maximal refined double character—denote the index set of those terms by $\tilde{L}_{\mu,*,k}$ (notice that the sublinear combination $\sum_{l\in L_{\mu,*,k}}$ will be included in those terms). We then derive that there exists a linear combination of acceptable $(\mu+1)$-tensor fields with a simple character $Cut[\vec{\kappa}_{simp}]$ so that:

$$\sum_{l\in\tilde{L}_{\mu,*,k}}a_l\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$-\sum_{h\in H}a_h X\,div_{i_{\mu+1}}C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$+\sum_{j\in J}a_jC^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,Y_1,Y_2,\phi_2,\dots,\hat{\phi}_k,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=0. \tag{A.18}$$

[74] Notice that (A.17) does not fall under any of the "forbidden cases", since the for the factor $\nabla^{(A)}Y$ we have $\Phi_1+M\geqslant 2$, where M is the number of free indices that belong to that factor and Φ_1 is the number of factors $\nabla\phi_h$ that contract against it.

Now, formally replacing the expression

$$\nabla^{(A)}_{y_1\dots y_A} Y_1 \otimes \nabla^{(B)}_{r_1\dots r_B} Y_2 \otimes \nabla^{y_q}\phi_z \otimes \cdots \otimes \nabla^{y_{x-1}}\phi_\chi \otimes \nabla^{y_x}\upsilon \dots \nabla^{y_A} \otimes \upsilon$$

by an expression

$$S_*\nabla^{(A-2)}_{y_1\dots y_{A-2}} R_{iy_{A-1}y_A l} \otimes S_*\nabla^{(B-2)}_{r_1\dots r_{B-2}} R_{i'r_{B-1}r_B}{}^{l} \otimes \nabla^i\phi_1 \otimes \nabla^{i'}\phi_k$$

and repeating the permutations that make the above hold formally,[75] we derive our claim. □

Appendix B. A graph-theoretical translation of a lemma of Alexakis (by Paul Christiano and Travis Schedler[76])

In this appendix, we give a graph-theoretical translation of an algebraic lemma of Alexakis ([2, Proposition 5.2]; see also the introduction of the present paper) which makes up the bulk of his proof of the Deser–Schwimmer conjecture. It is hoped that, by formulating it in this way, it will be possible to find a simplified combinatorial proof of this lemma, since Alexakis's proof, found in this paper and the next two in the series, takes hundreds of pages. (More precisely, in these three papers, Alexakis proves this lemma along with the slight generalizations [3, Propositions 3.1 and 3.2], collectively called the "main algebraic propositions".)

In order to prove the equivalence of Alexakis's lemma with our graph-theoretical statement, we also provide a more algebraic formulation of the former in Section B.2 below using the language of tensors over $\mathbb{R}$ as well as $\mathcal{O}_M$. This may be of independent interest.

B.1. Graph-theoretical statement

In this note, the term "graph" refers to an undirected graph with no loops, which is built out of *half-edges*: each half-edge is incident to a single vertex. A (full) edge is a pair of connected (distinct) half-edges, which must be incident to distinct vertices (since we require the graph to have no loops). Additionally, each half-edge can be in at most one full edge, but may also be part of no full edge. We call half-edges which are not part of a full edge *tails*.

Let $G_{p,s}$ be the set of isomorphism classes of partially-labeled graphs (with tails and without loops) with p white vertices and s black vertices, subject to the following valence conditions and labeling data. The white vertices are labeled 1 through s and are each incident to at least two half-edges, all of which are unlabeled. The black vertices are themselves unlabeled, and are each incident to four distinguished half-edges, labeled 1, 2, 3, and 4. Black vertices may also be incident to any number of unlabeled half-edges.

Let $\overline{\mathbb{Q}G_{p,s}}$ be the space of formal linear combinations of elements of $G_{p,s}$, modulo the following relations. For any graph Γ, any black vertex v, and any permutation $\tau \in S_4$, denote by $\tau_v\Gamma$ the graph obtained by applying τ to the labels of v. Then, for every graph $\Gamma \in G_{p,s}$ and every black vertex $v \in \Gamma$, we quotient by the relations $(12)_v\Gamma + \Gamma = 0$, $(34)_v\Gamma + \Gamma = 0$, and $(123)_v\Gamma + (132)_v\Gamma + \Gamma = 0$. Furthermore, if v has any unlabeled half-edges, and $\tilde{\Gamma}$ is any graph

[75] See the argument or the proof of (4.22).

[76] This document is based on explanations and work of S. Alexakis. We kindly thank him for this. We are also grateful to P. Etingof and J. Steinhardt for useful discussions. This project was supported by the MIT Undergraduate Research Opportunities Program.

obtained from Γ by assigning one of these half-edges the label 5 (therefore v now has 5 rather than 4 labeled half-edges), then we quotient by $\mathrm{forget}_{v,5}((125)_v\tilde{\Gamma} + (152)_v\tilde{\Gamma} + \tilde{\Gamma}) = 0$, where $\mathrm{forget}_{v,5}$ forgets the label 5 at the vertex v.

For any tail h of a graph $\Gamma \in G_{p,s}$ we may define $D_h(\Gamma)$ to be the sum, over all vertices not incident to h, of the graph obtained by adding a new unlabeled tail at that vertex and connecting the new tail to h. Furthermore, let $D(\Gamma)$ be the linear combination of graphs *without tails* obtained by applying D_h successively to all tails h. It is easy to see that the result does not depend on the order in which this is done, and that the linear extension of this operation descends to $\overline{\mathbb{Q}G_{p,s}}$.

Alexakis' lemma is equivalent to the following graph-theoretical statement:

Lemma B.1. *Fix $p, s \geqslant 0$ with $p + s \geqslant 3$ and $X, Y \in \overline{\mathbb{Q}G_{p,s}}$ which are linear combinations of graphs with μ and $\mu + 1$ tails respectively. If $DX = DY$, then there exist graphs $\Gamma_i \in G_{p,s}$, each with $\mu + 1$ tails, specified tails h_i of each Γ_i, and coefficients $\lambda_i \in \mathbb{Q}$ such that $X = \sum \lambda_i D_{h_i}(\Gamma_i)$.*

We remark that the case $s = 0$ (no black vertices) is already highly nontrivial, and might be a good first case to consider in the search for a simple combinatorial proof.

B.2. Equivalence with Alexakis's lemma

Fix a smooth manifold M and a Riemannian metric g on M. We will henceforth refer to pairs (M, g) as *manifolds*, omitting "Riemannian". Let $W_M^k := W_M^{\otimes_{\mathcal{O}_M} k}$ be the space of cotensor fields of rank k, i.e., the kth tensor power of $\mathrm{Vect}(M)^* = W_M$ as a vector bundle (where we identify vector bundles with their spaces of global sections). *Caution*: this does not mean the exterior power, so that W_M^k is a vector bundle of rank $k \cdot \dim M$. We use W_M^k rather than $W_M^{\otimes k}$ because we reserve $\otimes := \otimes_{\mathbb{R}}$ to denote tensoring over the ground field $\mathbb{R}$. That is, $W_M^j \otimes W_M^k := W_M^j \otimes_{\mathbb{R}} W_m^k$ is a tensor product of the two infinite-dimensional vector spaces W_M^j and W_M^k, resulting in tensor products over $\mathbb{R}$ of rank j and rank k cotensor fields, as distinct from rank $j + k$ cotensor fields. To pass back to ordinary cotensor fields, we will use the multiplication map, $(W_M^j \otimes W_M^k) \twoheadrightarrow (W_M^j \otimes_{\mathcal{O}_M} W_M^k) = W_M^{j+k}$.

We will be interested in elements of vector spaces of the form

$$W_M^{c_1} \otimes \cdots \otimes W_M^{c_\sigma}, \tag{B.1}$$

where, we emphasize again, $\otimes := \otimes_{\mathbb{R}}$, *not* the tensor product as vector bundles, so the above elements are *not* cotensor fields but elements of a tensor product over $\mathbb{R}$ of cotensor fields. We will call each $W_M^{c_i}$ a *factor*, so that (B.1) has σ factors, and inside each factor $W_M^{c_i}$ we will refer to the c_i (*free*) *indices* of the cotensor fields.[77] Let $\nabla : W_M^k \to W_M^{k+1}$ be the covariant derivative, where the new index is the last one. We can extend this to a map

[77] Here, the adjective "free" is in accordance with the terminology of Alexakis: when we consider operators of the form $\mathcal{O}_M^{\otimes p} \to W_M^{c_1} \otimes \cdots \otimes W_M^{c_\sigma}$, defined as a composition of some operator valued in $W_M^{c_1'} \otimes \cdots \otimes W_M^{c_\sigma'}$ for possibly larger values $c_1' \geqslant c_1, \ldots, c_\sigma' \geqslant c_\sigma$, with some contractions against the Riemannian metric, then one can refer to all $c_1' + \cdots + c_\sigma'$ indices, with the $c_1 + \cdots + c_\sigma$ remaining ones after contraction called "free" and the others called "non-free" (or "contracted"). We will not need to refer to non-free indices, but rather only to free indices.

$$(\nabla)_i : W_M^{c_1} \otimes \cdots \otimes W_M^{c_i} \otimes \cdots \otimes W_M^{c_\sigma} \to W_M^{c_1} \otimes \cdots \otimes W_M^{c_i+1} \otimes \cdots \otimes W_M^{c_\sigma},$$

obtained by applying ∇ to the ith factor.

We can use g to define the maps

$$\begin{aligned}\mathrm{contr}^{(i_1,j_1),(i_2,j_2)} &: W_M^{c_1} \otimes \cdots \otimes W_M^{c_{i_1}} \otimes \cdots \otimes W_M^{c_{i_2}} \otimes \cdots \otimes W_M^{c_\sigma} \\ &\to W_M^{c_1} \otimes \cdots \otimes W_M^{c_{i_1}-1} \otimes \cdots \otimes W_M^{c_{i_2}-1} \otimes \cdots \otimes W_M^{c_\sigma},\end{aligned}$$

by contracting the j_1th (free) index of the i_1th factor with the j_2th (free) index of the i_2th factor using g. For notational convenience, we define $\mathrm{contr}^{(i_1,j_1),(i_2,j_2)}$ to be 0 when $i_1, i_2, j_1, j_2 \leqslant 0$, $i_i, i_2 > \sigma$, $j_1 > c_{i_1}$, or $j_2 > c_{i_2}$.

To prove the equivalence of Alexakis's lemma with Lemma B.1 above, we will develop a correspondence between certain linear operators $\mathcal{O}_M^{\otimes p} \to W_M^{c_1} \otimes \cdots \otimes W_M^{c_p} \otimes W_M^{d_1} \otimes \cdots \otimes W_M^{d_s}$ and certain graphs (which are either graphs in $G_{p,s}$ as above, or such graphs which are additionally equipped with labelings of some of the previously unlabeled vertices and half-edges). To begin, the identity map (for $s = 0$ and $c_i = 0$ for all i) is represented by a graph with p labeled vertices and no half-edges. Applying $(\nabla)_i$ to an expression corresponds to adding a *labeled* tail at vertex i. Applying $\mathrm{contr}^{(i_1,j_1),(i_2,j_2)}$ corresponds to connecting the j_1th tail of vertex i_1 to the j_2th tail of vertex i_2.

In particular, we are interested in the space $\mathcal{B}_{p,s}$ spanned by operators of the form

$$\begin{aligned}\Omega_1 \otimes \cdots \otimes \Omega_p \mapsto{}& \mathrm{contr}^{(i_1,j_1),(i'_1,j'_1)} \circ \cdots \circ \mathrm{contr}^{(i_\ell,j_\ell),(i'_\ell,j'_\ell)} \\ & \big(\big(\nabla^{(c_1)}\Omega_1\big) \otimes \cdots \otimes \big(\nabla^{(c_p)}\Omega_p\big) \otimes \big(\nabla^{(d_1)}R\big) \otimes \cdots \otimes \big(\nabla^{(d_s)}R\big)\big) \qquad \text{(B.2)}\end{aligned}$$

where R is the Riemann curvature tensor for the manifold, each $c_i \geqslant 2$, and $p + s \geqslant 3$. Given a manifold (M, g) and $\Omega_1, \ldots, \Omega_p \in \mathcal{O}_M$, we can evaluate any expression in $\mathcal{B}_{p,s}$ to obtain an element of the form (B.1) with $p + s = \sigma$. The *tensor rank* of such an expression is $\sum c_i + \sum d_i + 4s - 2\ell$. The *weight* of such an expression is $-(\sum c_i + \sum d_i + 2s)$.

Such an operator corresponds to a graph where there are s labeled white vertices, the ith one incident to c_i labeled half-edges, and t labeled black vertices, the jth one incident to $d_j + 4$ labeled half-edges. Moreover, the j_mth half-edge incident to vertex i_m is connected to the j'_mth half-edge incident to vertex i'_m for each $m \leqslant \ell$, to form ℓ full edges. The tensor rank is the number of tails, and the weight is $-\#(\text{half-edges}) + 2\#(\text{black vertices})$. Note that, for now, *all vertices and half-edges are labeled*. This will change when we perform a certain quotient of $\mathcal{B}_{p,s}$.

In general, given arbitrary $\alpha \in W_M^r$, the cotensor $\nabla^{(k)}\alpha$ is not symmetric in the new k indices. Skewsymmetrizing two of these new indices yields a new cotensor field which is a linear combination of terms of the form

$$\mathrm{contr}^{i,j}\big(\nabla^{(k')}\alpha \otimes_{\mathcal{O}_M} \nabla^{(k-2-k')}R\big), \qquad \text{(B.3)}$$

for some indices i and j, and some $k' \leqslant k - 2$, by the definition of R. Here, as there is only one factor (since we tensor over $\mathcal{O}_M$), $\mathrm{contr}^{i,j} := \mathrm{contr}^{(1,i),(1,j)}$.

Thus, define $\mathcal{E}_{p,s+1}$ to be the span of elements obtained from (B.2) by replacing an expression of the form $\alpha = \nabla^{(k)}\Omega_i$ or $\nabla^{(k)}R$ by a term of the form (B.3), or by iterating such substitutions.

Then, in the quotient, $\mathcal{B}_{p,s}/\mathcal{E}_{p,s+1}$, we can view most of the free indices in each given factor as symmetric. In particular, we can view all of the free indices of the first p factors of an element of (B.2), and the first d_i free indices of the last s factors, as symmetric.

Graphically, passing to symmetric tensors in this way corresponds to removing the labels of all the half-edges except for four of them at each black vertex (ones corresponding to R); we will label these 1, 2, 3, and 4, in order.

Next, Alexakis defined the following operator, "$\mathrm{XDiv}_{(i,j)}$". Applied to an element of $W_M^{c_1} \otimes \cdots \otimes W_M^{c_{i_1}} \otimes \cdots \otimes W_M^{c_{i_2}} \otimes \cdots \otimes W_M^{c_\sigma}$, it is defined as

$$\mathrm{XDiv}_{(i,j)} = \sum_{i' \neq i} \mathrm{contr}^{(i,j),(i',c_{i'}+1)} \circ (\nabla)_{i'}.^{78}$$

One can view $\mathrm{XDiv}_{(i,j)}$ as a transformation on operators of the form (B.2) by composition (i.e., by applying $\mathrm{XDiv}_{(i,j)}$ to the output of the operator). Graphically, $\mathrm{XDiv}_{(i,j)}$ then corresponds to D_h, where h is the tail corresponding to (i, j).

Let us define the "total XDiv" operator XDiv on an expression of the form (B.2) to be the result of applying $\mathrm{XDiv}_{(i,j)}$ successively to all (free) indices; in total, this means we apply $\mathrm{XDiv}_{(i,j)}$ a number of times equal to the tensor rank. Graphically, this corresponds to the operation D.

Finally, define the multiplication operation (replacing all of the $\otimes = \otimes_{\mathbb{R}}$ symbols by $\otimes_{\mathcal{O}_\mathcal{M}}$ except for the one between $W_M^{c_p}$ and $W_M^{d_1+4}$):

$$\mathrm{mult}_{p,s} : W_M^{c_1} \otimes \cdots \otimes W_M^{c_p} \otimes W_M^{d_1+4} \otimes \cdots \otimes W_M^{d_s+4} \to W_M^{\sum c_i} \otimes W_M^{\sum d_i + 4s}.$$

Graphically, applying $\mathrm{mult}_{p,s}$ has the effect of forgetting the labels of the black vertices. This doesn't affect the white vertices, which remain labeled since $\Omega_1, \Omega_2, \ldots, \Omega_p$ are independent input functions.

We can now state Alexakis' lemma:

Lemma B.4. *Fix $\mu, p, s \in \mathbb{N}$ and $F, F' \in \mathcal{B}_{p,s}$ with tensor rank μ and $\mu + 1$ respectively, and of weights $-n + \mu$, and $-n + \mu + 1$, respectively. If there exists an $\epsilon_1 \in \mathcal{E}_{p,s+1}$ such that*

$$\mathrm{mult}_{p,s} \mathrm{XDiv}(F) = \mathrm{mult}_{p,s}\big(\mathrm{XDiv}\big(F'\big) + \epsilon_1\big) \tag{B.5}$$

for every manifold (M, g) of dimension $\dim M = n$, and all $\Omega_1, \ldots, \Omega_p \in \mathcal{O}_M$, then there exists $F_\ell \in \mathcal{B}_{p,s}$, certain free indices (i_ℓ, j_ℓ) thereof, and some $\epsilon_2 \in \mathcal{E}_{p,s+1}$ such that

$$\mathrm{mult}_{p,s} F = \mathrm{mult}_{p,s}\left(\sum_i \mathrm{XDiv}_{(i_\ell, j_\ell)}(F_\ell) + \epsilon_2\right) \tag{B.6}$$

for every such (M, g) and all $\Omega_1, \ldots, \Omega_p \in \mathcal{O}_M$.

Sketch of proof of equivalence of Lemmas B.1 and B.4. One direction is straightforward: to any graph Γ with s labeled white vertices and t labeled black vertices, and such that four of the half-edges incident to each black vertex are labeled by 1, 2, 3, and 4, we can associate an operator

[78] The reason for the $c_{i'} + 1$ superscript is our convention that the new index in $\nabla(\alpha)$ is the last one.

$\theta_\Gamma \in \mathcal{B}_{p,s}/\mathcal{E}_{p,s+1}$. The resulting element $\operatorname{mult}_{p,s}\theta_\Gamma$ only depends on the graph obtained from Γ by forgetting the labeling of the black vertices.

Moreover, this extends to linear combinations of graphs. We claim that the result depends only on the class in $\overline{\mathbb{Q}G_{p,s}}$, and yields a map $\overline{\mathbb{Q}G_{p,s}} \to \operatorname{mult}_{p,s}(\mathcal{B}_{p,s}/\mathcal{E}_{p,s+1})$. To see this, it suffices to compare the symmetry relations of Section B.1 with the symmetry conditions that R satisfies. Let us recall these symmetries.

If τX denotes the result of applying τ to the components of X, R always satisfies the identities

$$(12)R = -R, \tag{B.7}$$

$$(34)R = -R, \tag{B.8}$$

$$R + (123)R + (132)R = 0, \tag{B.9}$$

and ∇R satisfies the identity

$$\nabla R + (125)\nabla R + (152)\nabla R = 0. \tag{B.10}$$

Moreover, the same identities are satisfied if we substitute $\nabla^{(k)}R$ for R; we may also allow 5 to be any index greater than 4 if we work modulo terms involving two copies of R (i.e., of the form (B.3) where $\alpha = \nabla R$). Since these match the defining relations of $\overline{\mathbb{Q}G_{p,s}}$, the map descends as claimed.

The harder part of the equivalence is to show that this map from graphs to operators is injective. To do so, we use the second fundamental theorem of invariant theory for $O(V)$. First, let us evaluate operators in $\operatorname{mult}_{p,s}(\mathcal{B}_{p,s}/\mathcal{E}_{p,s+1})$ at a fixed $x \in M$ to obtain constant-coefficient differential operators. Set $V := \mathcal{T}_x M$, a vector space equipped with an inner product from the Riemannian metric, and hence an action of $O(V)$. Thus, $V^* \cong W_{M,x}$, the cotangent space at x. Furthermore, let $U \subseteq V^{\otimes 4}$ be the subspace satisfying the symmetry conditions (B.7)–(B.9), which is an irreducible representation of S_4 corresponding to the Young diagram $(2,2)$. We consider U^* as a subspace of $(V^*)^{\otimes 4}$, and it contains the evaluation of any curvature tensor R at x.

Under this evaluation, the first-order differential operation $\Omega \mapsto \nabla\Omega$ evaluates to the canonical element $\iota \in V \otimes V^*$ corresponding to the identity map $V^* \to V^*$. Similarly, replacing an operation θ with a contraction $\operatorname{contr}^{(i,j)}\theta$ corresponds to applying the pairing $V^* \otimes V^* \to \mathbb{R}$ inverse to the inner product on V.

Hence, we can identify operators $\operatorname{mult}_{p,s}(\mathcal{B}_{p,s}/\mathcal{E}_{p,s+1})$, evaluated at x, with certain $O(V)$-invariant elements of $TV \otimes TV^*$, satisfying certain symmetry conditions.

In more detail, an operator of the form (B.2) is homogeneous under the bigrading by number of copies of V and V^*. It has $\sum_i c_i + \sum_i d_i + 4s$ copies of V, and the number of copies of V^* is the tensor rank of the image of the operator.

By the second fundamental theorem of invariant theory [9], in the formulation of [19, Theorem B.2],[79] to show that no nontrivial relations arise, it suffices to show that the total number of rows which appear in the Young diagrams for the $S_{(\sum_i c_i + \sum_i d_i + 4s)}$ action by permuting the copies of V does not exceed $\dim V = n$, and similarly for the copies of V^*.

[79] This theorem is also used routinely in Alexakis's paper to show that certain equations defined for a particular dimension $n = \dim M$ hold "formally".

Let us restrict our attention to graphs in $G_{p,s}$ with μ tails, i.e., such that the image of $\mathrm{mult}_{p,s} \circ \theta_\Gamma$ has tensor rank μ. By assumption and the definition of weight, $n = \sum_i c_i + \sum_i d_i + 2s + \mu$, where μ is the tensor rank of the image of $\mathrm{mult}_{p,s} \circ \theta_\Gamma$. Since U occurs s times and is an irreducible representation of S_4 with only two rows, the desired statement follows for the $S_{(\sum_i c_i + \sum_i d_i + 4s)}$ action permuting the copies of V. On the other hand, $n \geqslant \mu$, so the desired statement also follows for the S_μ action permuting the copies of V^*. Thus, no nontrivial relations arise, and the map $\overline{\mathbb{Q}G_{p,s}} \to \mathrm{mult}_{p,s}(\mathcal{B}_{p,s}/\mathcal{E}_{p,s+1})$ is injective. $\square$

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